Systems of linear equations

System of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots&\ldots\ldots\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

where a_{ij}, b_i for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$
are given real numbers
and $x_1, x_2, \ldots, x_n$ are unknowns,
is called system of m linear equations with n unknowns.
Numbers a_{ij} are coefficients of the system,
numbers b_i are right hand-side terms.

System of linear equations can be symbolically written in matrix form

$$\mathbf{A} . \mathbf{X} = \mathbf{B}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ .. & .. & \cdots & .. \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

Matrix **A** is called matrix of the system.

System of equations is said to be

1. homogeneous (without right hand-side terms), if $b_i = 0$ for all $i = 1, 2, ..., m$
2. non-homogeneous (with right hand-side terms), if $b_i \neq 0$ for at least one i.

Solution of system of equations is any such n-tuple (ordered group) of real numbers $(r_1, r_2, ..., r_n)$, which being substituted to the system instead of unknowns x_i in the given order form m correct number equalities.

To solve the given system of equations means to perform the following steps of solution:

1. to find out whether the given system can have/has a solution
2. to determine number of possible solutions of the system
3. to find all solutions of the system.

- System of m linear equations with n unknowns can have:

1. exactly one solution
2. infinitely many solutions
3. no solution

- Homogeneous system of equations has always a solution, n-tuple $(0, 0, ..., 0)$, which is called trivial solution.
- If $m < n$, homogeneous system of linear equations has infinitely many solutions.

- Two systems of linear equations are called equivalent, if any solution of one of them is also solution of the other one.

- Equivalent manipulations with the system of equations are such manipulations, which do not change the set of all solutions of the given system.

1. Change of order of system equations.
2. Multiplication (division) of arbitrary system equation by a non-zero number $c \neq 0$.
3. Addition of a non-zero multiple of one equation (or multiples of several equations) of the system to another one (or to other ones) from the system.
4. Elimination of equation, which is multiple of another equation (or combination of other equations) of the system.

Gauss elimination method

Method to solve linear systems of equations
by consecutive elimination of unknowns.
Matrix **A** of the system, or extended matrix of the system

$$\mathbf{A}' = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

are transformed by means of equivalent manipulations
into the triangular form.

Any solution of the system determined by obtained triangular matrix is also solution of the original system.

Frobenius theorem. System of non-homogeneous equations is solvable if and only if the rank of the matrix of the system equals to the rank of the extended matrix of the system.

Consequence 1. If $h(\mathbf{A}) = h(\mathbf{A}') = n$ (n is number of unknowns), then system has exactly one solution.

Consequence 2. If $h(\mathbf{A}) = h(\mathbf{A}') < n$ (n is number of unknowns), then system has infinitely many solutions and $n - h$ unknowns can be chosen arbitrary.

Consequence 3. If $h(\mathbf{A}) \neq h(\mathbf{A}')$, then system has no solution.