LIMIT OF A FUNCTION

δ-neighbourhood of the point *a* is an interval in *R*

$$O_{\partial}(a) = (a - \partial, a + \partial),\ \partial > 0$$

Right-hand neighbourhood of the point *a*

$$O_{\partial}^{+}(a) = (a, a + \partial),\ \partial > 0$$

Left-hand neighbourhood of the point *a*

$$O_{\partial}^{-}(a) = (a - \partial, a),\ \partial > 0$$

Heine definition of the limit of a function

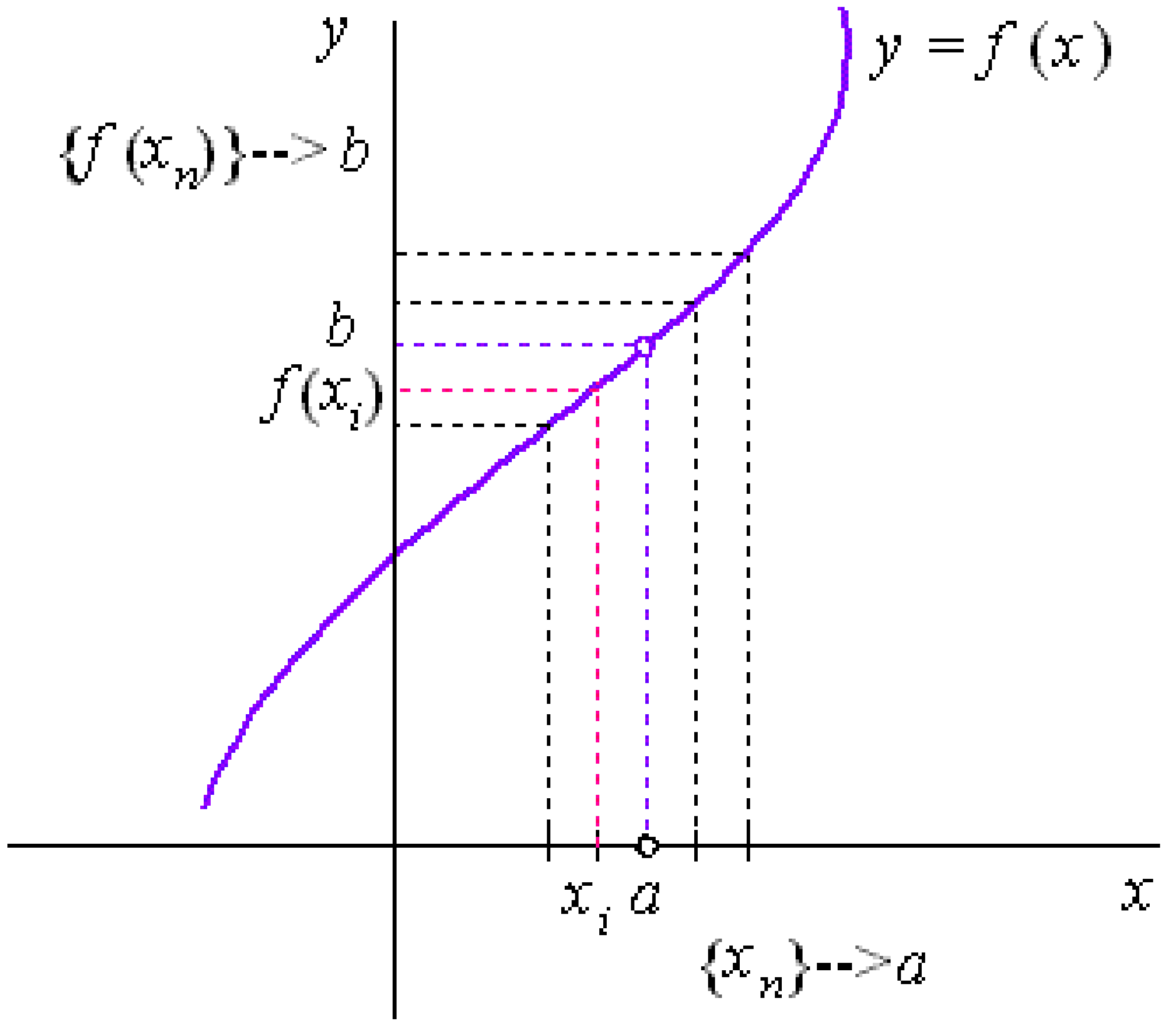
Let f be a function defined for all $x \neq a$ on some neighbourhood $O_\delta(a)$ of the point a. Function f is said to have a limit equal to b at the point a if for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in D(f)$, $x_n \neq a$, $\lim\limits_{n\to\infty} x_n = a$

sequence of function values $\{f(x_n)\}_{n=1}^{\infty}$

has a limit equal to number b, $\lim\limits_{n\to\infty} f(x_n) = b$.

$$\lim_{x\to a} f(x) = b \Leftrightarrow$$

$$(\forall \{x_n\}_{n=1}^{\infty}, x_n \in D(f), x_n \neq a, \lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = b)$$



y
$y = f(x)$
$\{f(x_n)\} \dashrightarrow b$
b
$f(x_i)$
x_i
a
x
$\{x_n\} \dashrightarrow a$

From the definition follows that the limit

$$\lim_{x \to a} f(x) = b$$

does not exist, if we can find two sequences

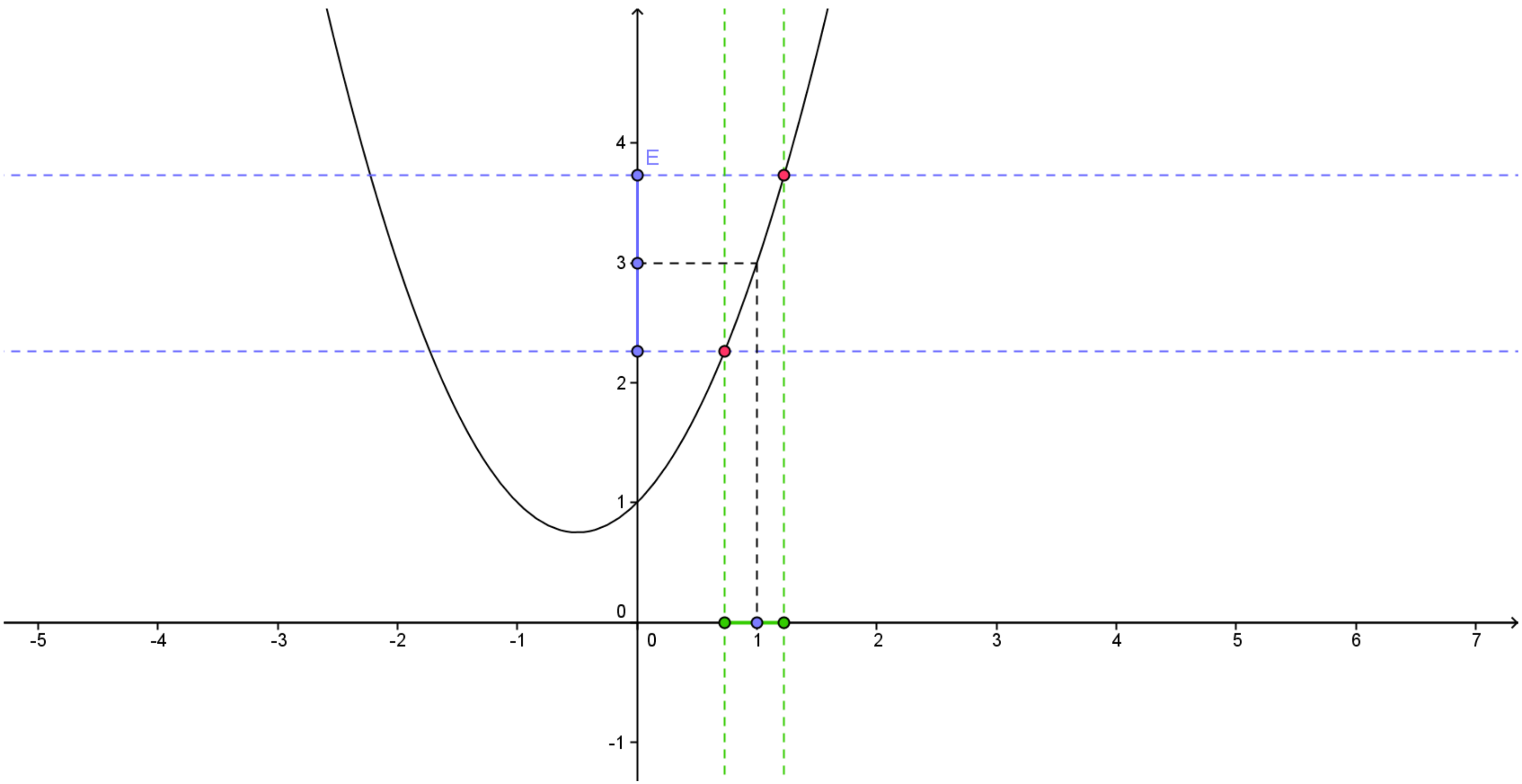
$$\{x_n\}_{n=1}^{\infty} \qquad \{x_n'\}_{n=1}^{\infty}$$

such that

$$x_n, x_n' \in D(f), x_n \neq a, x_n' \neq a, \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n' = a$$

while $\displaystyle \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(x_n')$

$$\lim_{x \to 0} \cos\frac{1}{x}, \textit{does not exist} \qquad \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$



Cauchy definition of a limit

Let f be a function defined for all $x \neq a$ on some neighbourhood $O_\delta(a)$ of the point a. Function f is said to have the limit equal to the number b at the point a,

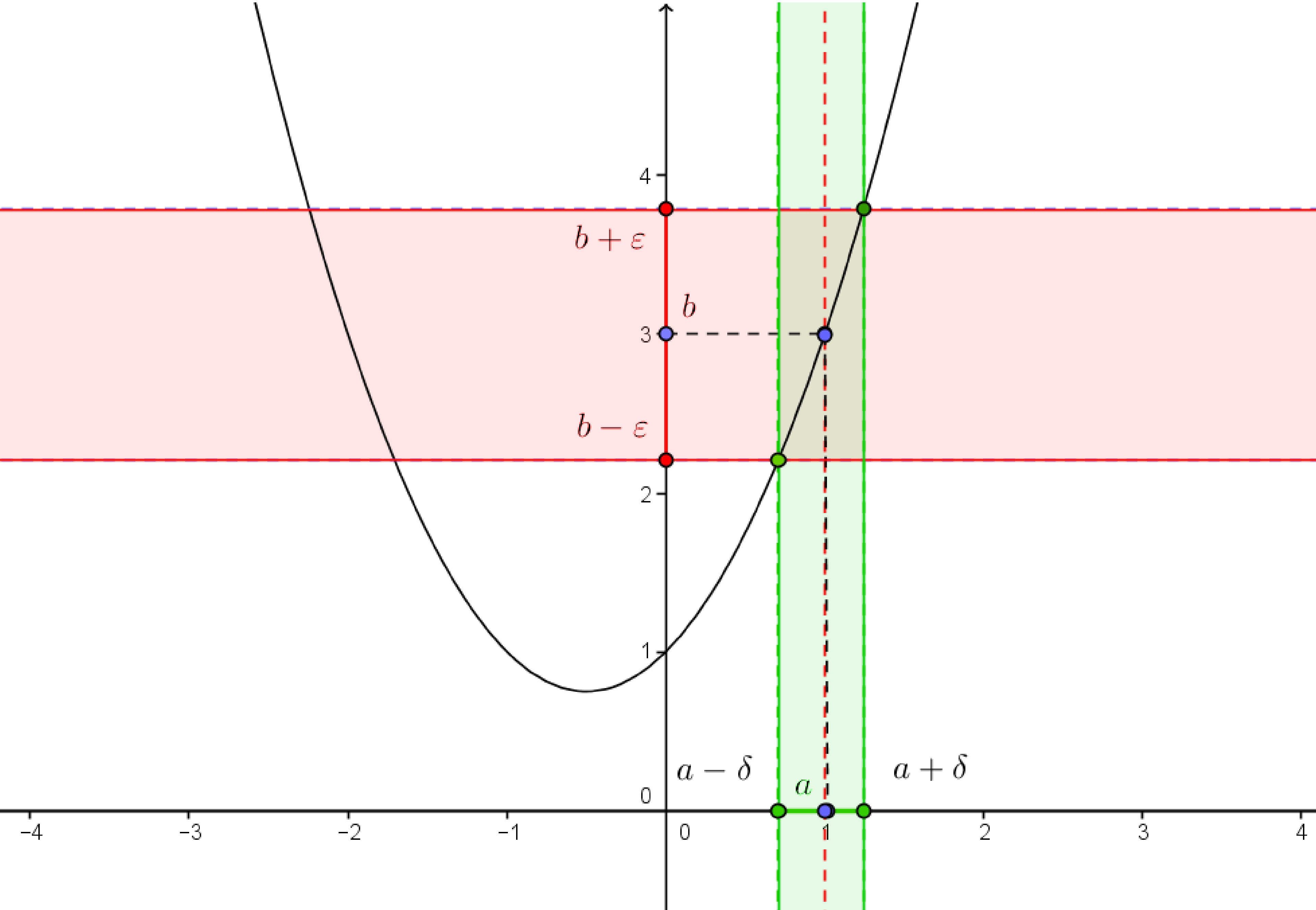
if for any real number $\varepsilon > 0$ there exists such real number $\delta > 0$, that for all $x \in O_\delta(a)$, $x \neq a$ is $f(x) \in O_\varepsilon(b)$.

$$\lim_{x \to a} f(x) = b \Leftrightarrow$$

$$(\forall \varepsilon > 0, \exists \partial > 0, \forall x \in O_\partial(a), x \neq a \Rightarrow f(x) \in O_\varepsilon(b))$$

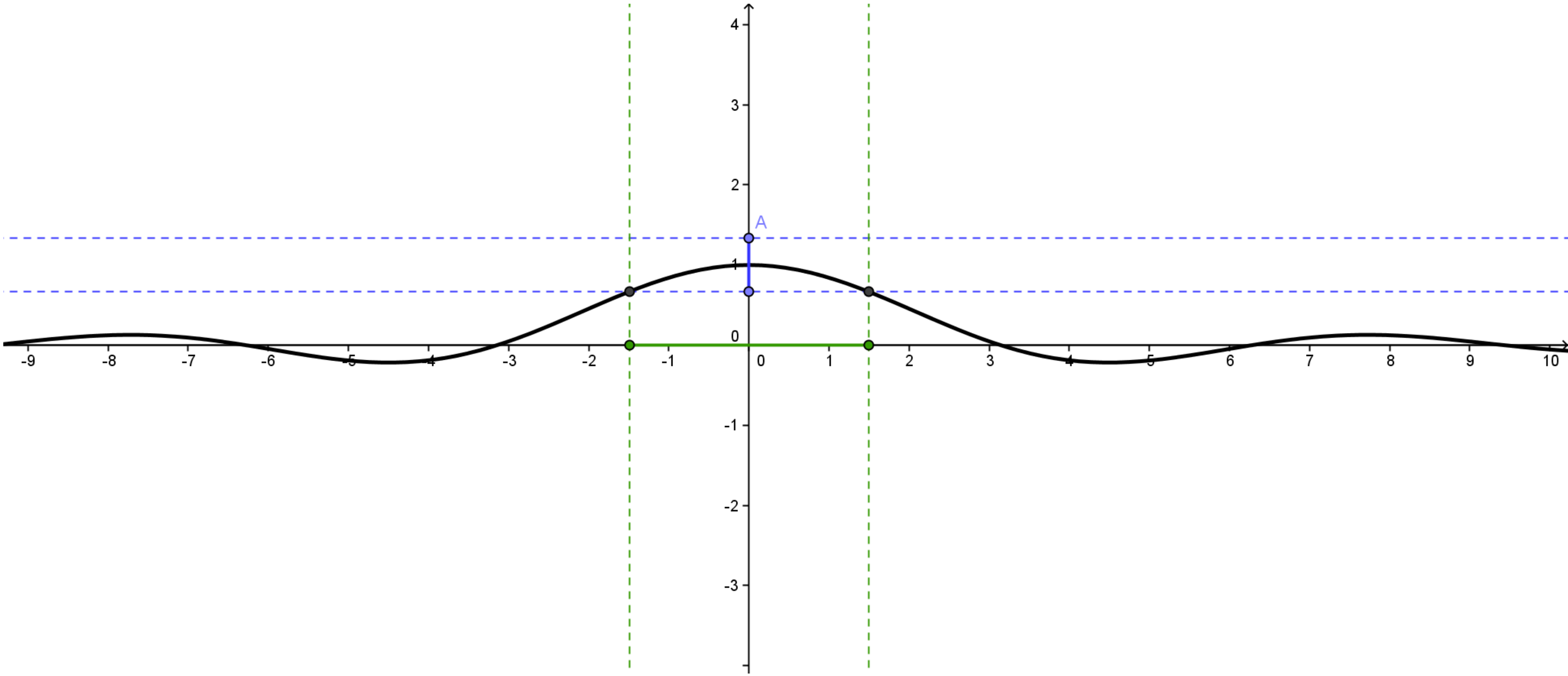
$$\lim_{x \to a} f(x) = b \Leftrightarrow$$

$$(\forall \varepsilon > 0, \exists \partial > 0, 0 < \left| x - a \right| < \partial \Rightarrow \left| f(x) - b \right| < \varepsilon)$$



$b+\varepsilon$
b
$b-\varepsilon$
$a-\delta$
a
$a+\delta$
4
3
2
1
0
-4
-3
-2
-1
0
1
2
3
4

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\ \lim_{x\to\pm\infty}\frac{\sin x}{x}=0$$



Function does not have the limit equal to number b at the point a, if at least for one ε-neighbourhood of b, $O_\varepsilon(b) = (b - \varepsilon, b + \varepsilon)$

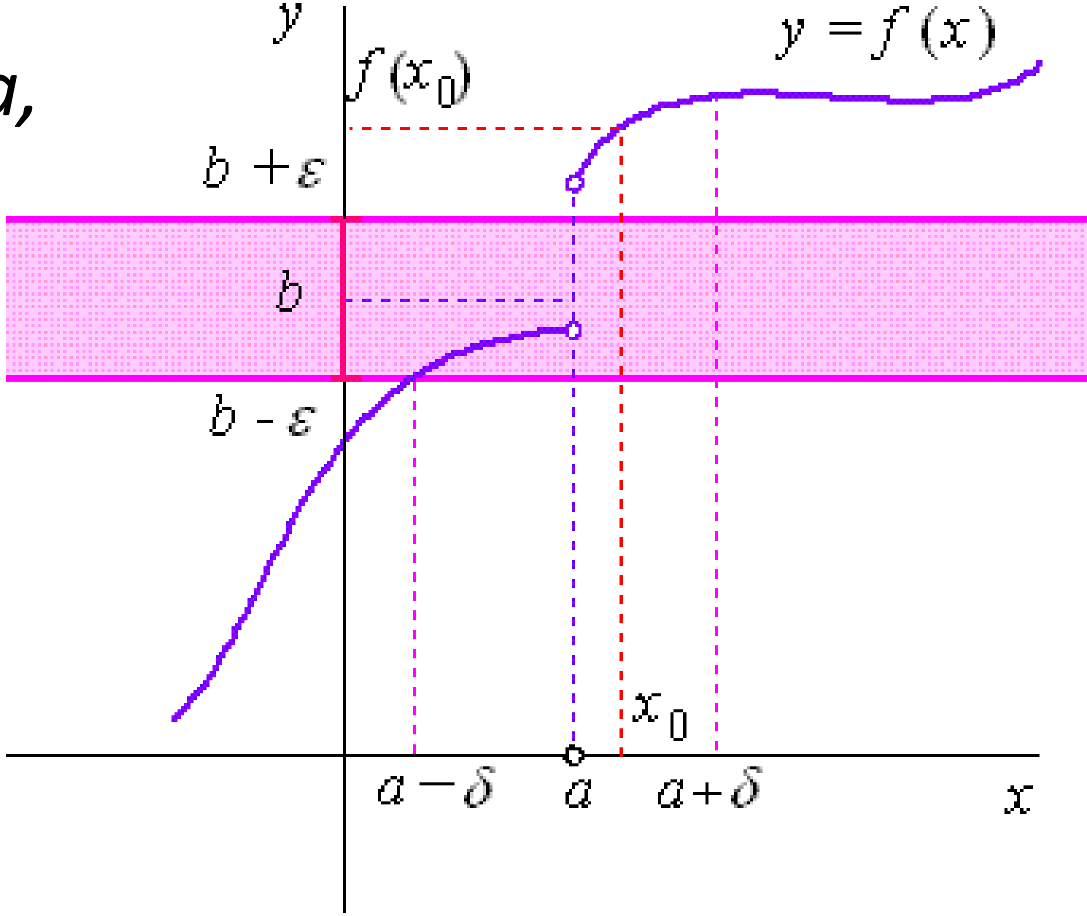
the required

δ-neighbourhood of a,

$O_\delta(a) = (a - \delta, a + \delta)$

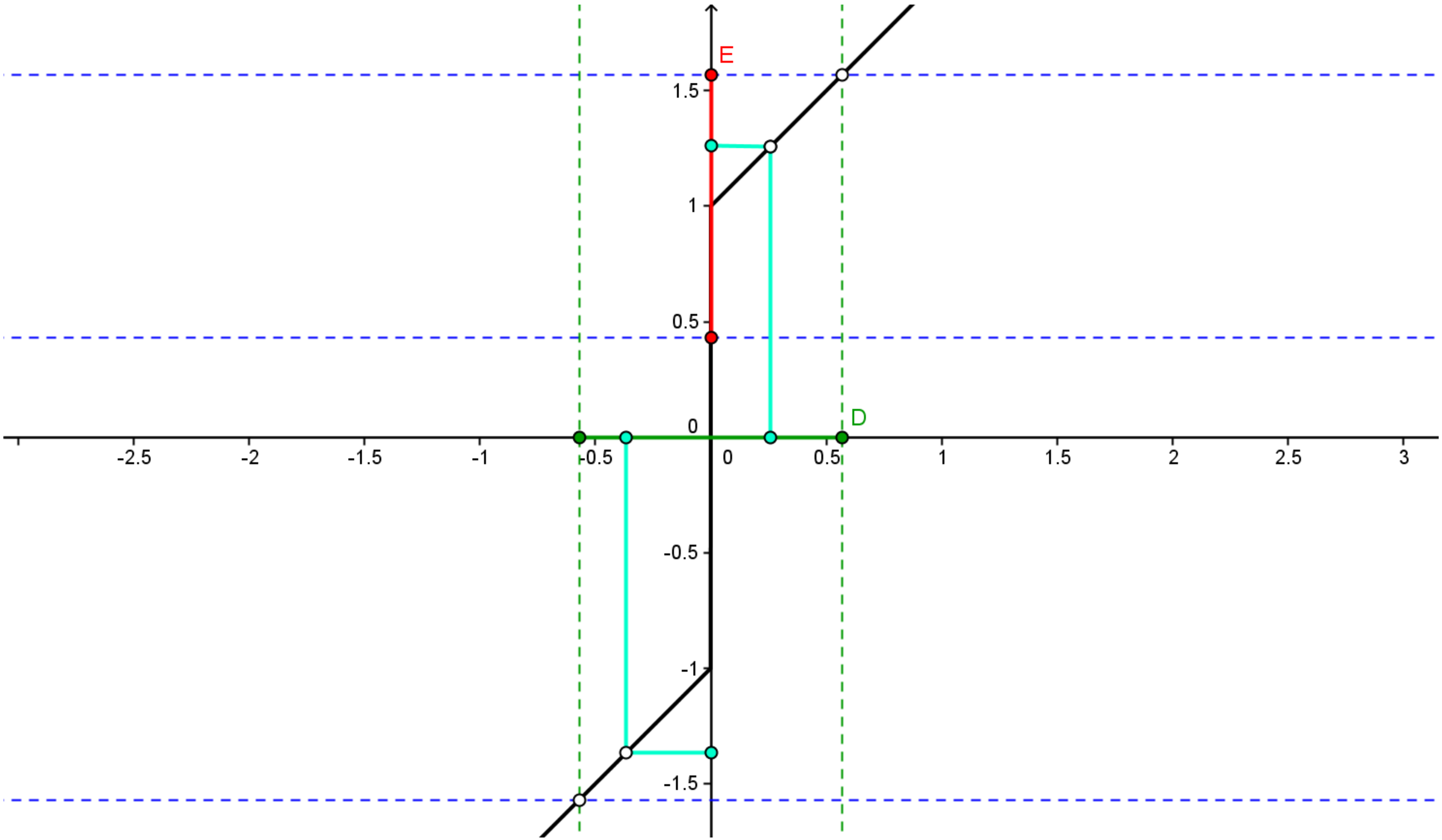
from the definition

does not exist.



$$\lim_{x\to 0}[\operatorname{sgn}(x)+x]\ \textit{does not exist}$$

$$\lim_{x\to 0^+}[\operatorname{sgn}(x)+x]=1,\ \lim_{x\to 0^-}[\operatorname{sgn}(x)+x]=-1$$



One-sided limit of a function – Heine definition

Let *f* be a function defined for all $x \neq a$ on some left-hand neighbourhood $O^{-}_{\delta}(a)$ of point *a*.

$$\lim_{x \to a^-} f(x) = b \Leftrightarrow$$

$$(\forall \{x_n\}_{n=1}^{\infty}, x_n < a, \lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = b)$$

Function *f* is said to have one-sided limit, number *b* is called limit on the left of *f* at *a*.

One-sided limit of a function – Cauchy definition

Let *f* be a function defined for all $x \neq a$ on some right-hand neighbourhood $O^{+}_{\delta}(a)$ of point *a*.

$$\lim_{x \to a^{+}} f(x) = b \Leftrightarrow$$

$$(\forall \varepsilon > 0, \exists \partial > 0, \forall x \in O_{\partial}^{+}(a) \Rightarrow f(x) \in O_{\varepsilon}(b))$$

Function *f* is said to have one-sided limit,

number *b* is called limit on the right of *f* at *a*.

Limit of a function *f* at the point *a* does exist if and only if there exist:

- limit on the right of *f* at *a*
- limit on the left of *f* at *a*

and these two one-sided limits are equal.

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

Basic properties of limits of functions

1. Any function at any point has at most one limit.
2. If there exists a limit of a function *f* at a point *a*, then the function *f* is bounded at the neighbourhood of the point *a*.
3. If $f(x) = c,\ c \in R$,

   then
$$\lim_{x \to a} f(x) = c \ \forall a \in R$$

4. If $\lim\limits_{x \to a} f(x) = A, \lim\limits_{x \to a} g(x) = B$, then

a) $\lim\limits_{x \to a}(f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = A \pm B$

b) $\lim\limits_{x \to a}(f(x).g(x)) = \lim\limits_{x \to a} f(x).\lim\limits_{x \to a} g(x) = A.B$

c) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{A}{B},$

$B \neq 0, \exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a, g(x) \neq 0$

d) $\lim\limits_{x \to a}(f(x))^{g(x)} = (\lim\limits_{x \to a} f(x))^{\lim\limits_{x \to a} g(x)} = A^B$

if $\exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a$

formulas $(f(x))^{g(x)}$ and A^B have a meaning

5. Limit of three functions

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = b$ and if

$\exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a, \quad f(x) \leq g(x) \leq h(x),$

then $\lim_{x\to a} g(x) = b$

6. If on some neighbourhood $O_{\delta}(a)$ of point a holds

$$f(x) = g(x)\ \forall x \neq a \text{ and } \lim_{x\to a} g(x) = b,$$

then

$$\lim_{x\to a} f(x) = b$$

Properties 1) – 6) hold also for one-sided limits.

Improper limit of a function at a point *a*

$$\lim_{x \to a} f(x) = \infty \Leftrightarrow$$

$$(\forall K \in R, \exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a \Rightarrow f(x) > K)$$

$$\lim_{x \to a} f(x) = -\infty \Leftrightarrow$$

$$(\forall K \in R, \exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a \Rightarrow f(x) < K)$$

Other properties of limit of a function at a point *a*

7. If $\lim\limits_{x\to a} f(x) = b, \lim\limits_{x\to a} g(x) = 0,$

$$\exists \partial > 0, \forall x \in O_{\partial}(a), x \neq a$$

then
$$\frac{f(x)}{g(x)} > 0 \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \infty$$

$$\frac{f(x)}{g(x)} < 0 \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = -\infty$$

8. Let function *f* be bounded on some neighbourhood $O_\delta(a)$ of the point *a* and $\lim\limits_{x \to a} g(x) = \pm\infty$, then

a) $\lim\limits_{x \to a} (f(x) + g(x)) = \pm\infty$

b) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$

Proper and improper limit of a function at infinity (at improper points)

Let f be a function defined on interval (a, ∞).

$$\lim_{x\to\infty} f(x) = b \Leftrightarrow$$

$$(\forall \varepsilon > 0, \exists A \in R : x > A \Rightarrow f(x) \in O_{\varepsilon}(b))$$

$$\lim_{x\to\infty} f(x) = \infty \Leftrightarrow$$

$$(\forall K \in R, \exists A \in R : x > A \Rightarrow f(x) > K)$$

$$\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow$$

$$(\forall K \in R, \exists A \in R : x > A \Rightarrow f(x) < K)$$

Let *f* be a function defined on interval (-∞, *a*).

$$\lim_{x\to-\infty} f(x)=b \Leftrightarrow$$

$$(\forall\varepsilon>0, \exists A\in R: x<A\Rightarrow f(x)\in O_{\varepsilon}(b))$$

$$\lim_{x\to-\infty} f(x)=\infty \Leftrightarrow$$

$$(\forall K\in R, \exists A\in R: x<A\Rightarrow f(x)>K)$$

$$\lim_{x\to-\infty} f(x)=-\infty \Leftrightarrow$$

$$(\forall K\in R, \exists A\in R: x<A\Rightarrow f(x)<K)$$