SEQUENCES

- Any function *f* defined on the set of natural numbers is called a sequence.
- Range of its values $H(f)$ can be any set, if it is a set of real numbers, $H(f) \subset \mathbf{R}$, it is called a numerical or number sequence.

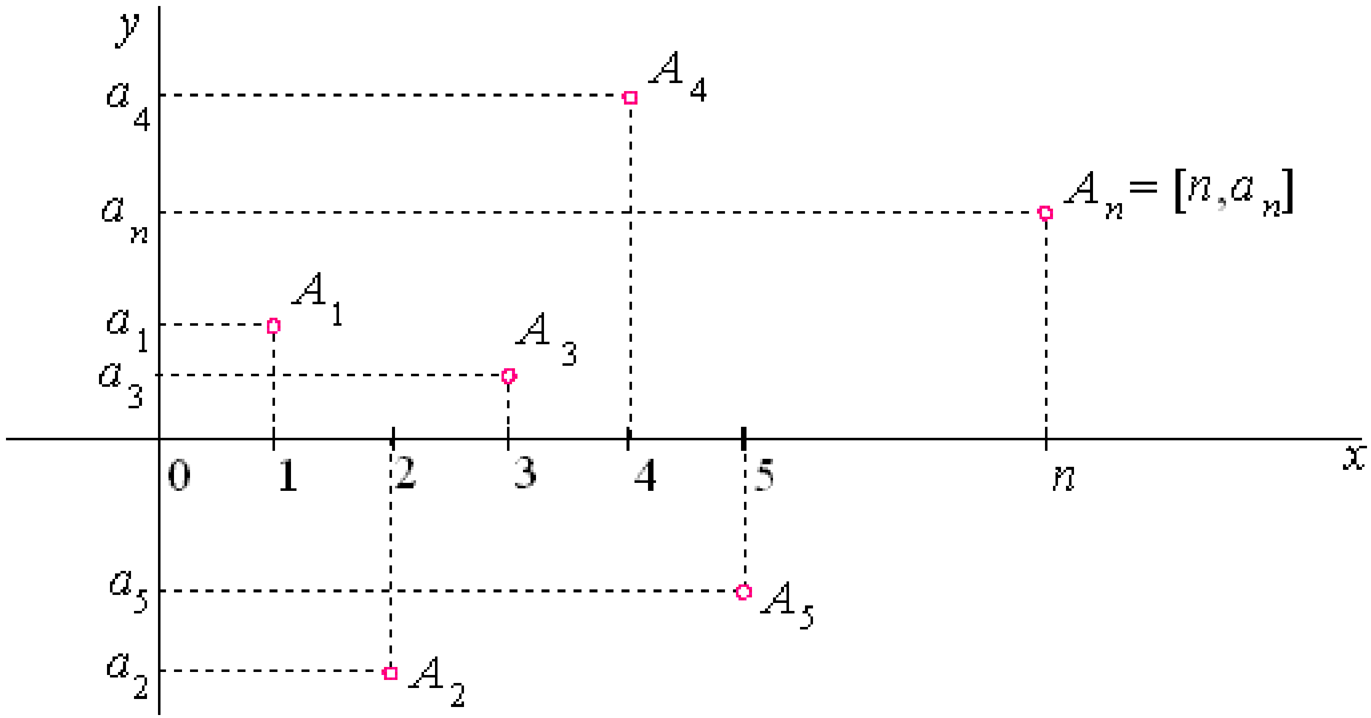
Denoting

- $f(1) = a_1, f(2) = a_2, \ldots, f(n) = a_n, \ldots$
  element $a_n = f(n)$ is called
  the *n*-th member or **term** of the sequence.
- Sequence is determined by its terms, we refer to it by denotation

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots, a_n, \ldots$$

Number sequence is a function defined on the set of all natural numbers, whose graph is the set of isolated points in the plane

$$\{A_n = [n, a_n], n \in N, a_n \in R\}$$



Properties of sequences

Sequence $\{a_n\}$ is

- increasing $\Leftrightarrow \forall\ n \in N, \quad a_n < a_{n+1}$
- non-decreasing $\Leftrightarrow \forall\ n \in N, a_n \leq a_{n+1}$
- decreasing $\Leftrightarrow \forall\ n \in N, \quad a_n > a_{n+1}$
- non-increasing $\Leftrightarrow \forall\ n \in N, \ a_n \geq a_{n+1}$

All above sequences are said to be monotone sequences.

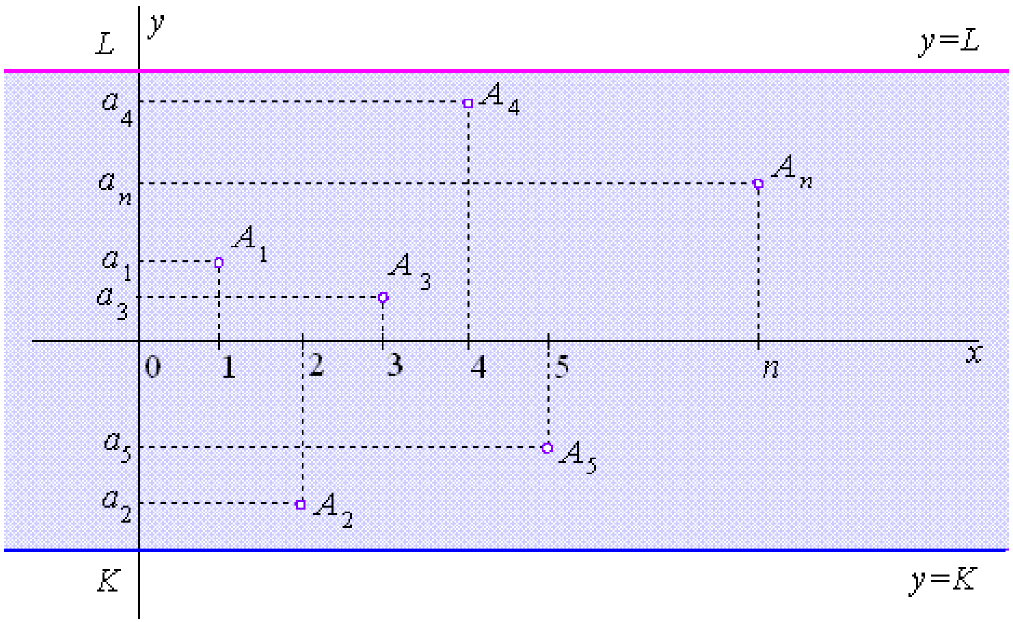
Increasing and decreasing sequences are said to be strictly monotone sequences.

Sequence $\{a_n\}$ is said to be

- bounded below $\Leftrightarrow \exists\, K \in R{:}\ \forall\, n \in N$ holds $K < a_n$
- bounded above $\Leftrightarrow \exists\, L \in R{:}\ \forall\, n \in N$ holds $a_n < L$

Sequence, which is both bounded above and bounded below, is said to be a bounded sequence.

All points on the graph of a bounded sequence are located in the layer between parallel lines with equations $y = K$, $y = L$.



Number *a* is called a limit of the sequence $\{a_n\}_{n=1}^{\infty}$, iff for any real number $\varepsilon > 0$ there exists such number n_0, that for all natural numbers $n > n_0$ holds

$$|a_n - a| < \varepsilon.$$

The fact that number *a* is a limit of the sequence $\{a_n\}$ is written as

$$\lim_{n\to\infty} a_n = a$$

The above definition of the limit of a sequence can be rewritten using quantifiers

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \forall \varepsilon > 0\, \exists n_0 : \forall n > n_0, n \in N : \left|a_n - a\right| < \varepsilon$$

Sequence with a proper limit is said to be **convergent**.
Sequence with no proper limit is said to be **divergent**.

If the limit of the sequence $\{a_n\}$ is a, we speak about sequence converging to the number a.

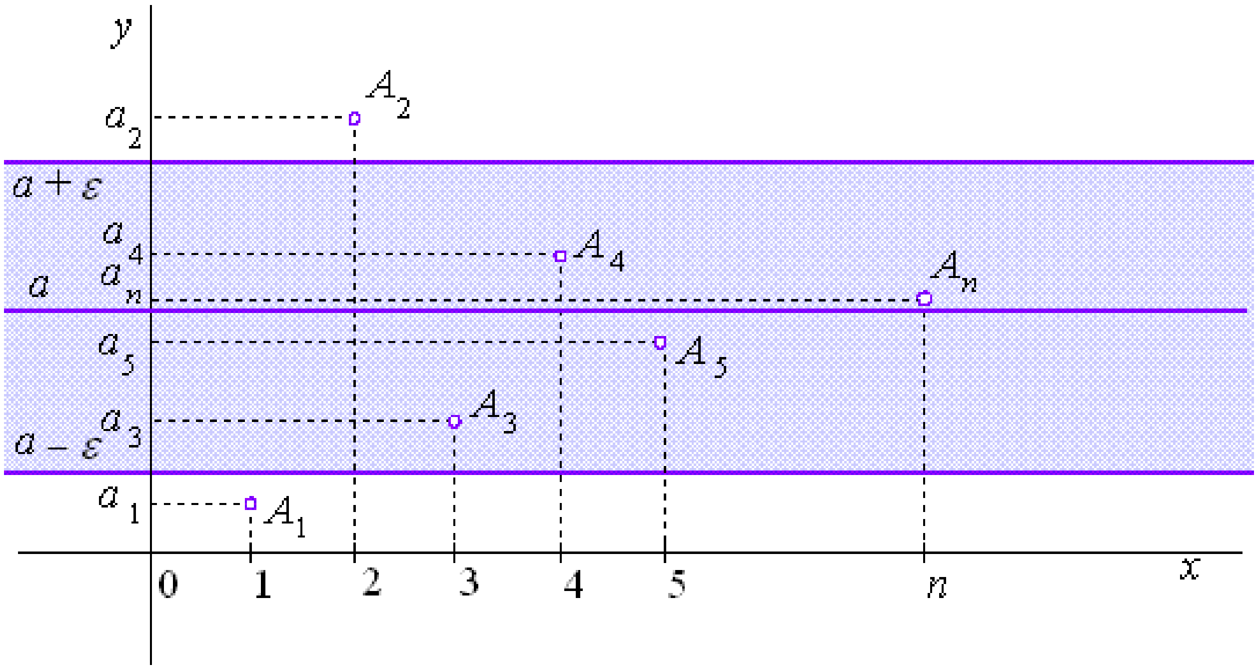
Properties

1. Any convergent sequence has a unique limit.

2. Any convergent sequence is bounded.

Limit and graph of a sequence

Geometric interpretation of the sequence convergence is, that most of the points A_n on the sequence graph are located in the layer determined by the parallels to the *x*-axis, passing through points $[0, a-\varepsilon]$ and $[0, a+\varepsilon]$ on the *y*-axis.



$$\lim_{n\to\infty} a_n = a$$

Number *a* is called the limit of the sequence $\{a_n\}$, iff almost all its terms are located in an arbitrary neighbourhood $O_\varepsilon(a)$.

Property of the sequence terms „to be close to the number, which is the limit of the sequence" holds for almost all terms of the convergent sequence, up to the finite number of them.

Number a is called the limit of the sequence $\{a_n\}$, if for any real number $\varepsilon > 0$ and for almost all terms of the sequence inequality $|a_n - a| < \varepsilon$ holds.

Number of those sequence terms, for which inequality $|a_n - a| < \varepsilon$ does not hold, depends on the choice of the positive number ε.
Decreasing the value of ε the number of these sequence terms is increasing, but for any ε it is a proper real number.

Subsequence of a sequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and $\{k_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers.

Sequence $\{a_{k_n}\}_{n=1}^{\infty}$ is called subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$ determined by the sequence $\{k_n\}_{n=1}^{\infty}$.

Subsequence of a sequence and its convergence

Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with the limit equal to number *a*.

Then any subsequence $\{a_{k_n}\}_{n=1}^{\infty}$ of this sequence is also convergent and its limit equals to the same number *a*.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{k_n} = a$$

Correspondence between convergence of a sequence and its subsequences is as follows.

1) If a sequence has a limit, then each its subsequence has the same limit.

2) If a sequence includes
a diverging subsequence or
2 subsequences converging to different limits,
then it is divergent.

Limit of three sequences

Let $\lim_{n\to\infty} a_n = l, \lim_{n\to\infty} c_n = l$

And let for any natural number *n* holds

$$a_n \leq b_n \leq c_n \ ,$$

then also

$$\lim_{n\to\infty} b_n = l \ .$$

Let $\{a_n\}$ and $\{b_n\}$ be two sequences

Sequences defined as

- $\{a_n + b_n\}, \{a_n - b_n\}, \{a_n . b_n\}, \{a_n/b_n\}$

are called sum, difference, product and quotient (for all $b_n \neq 0$) of the two sequences.

In case of two convergent sequences $\{a_n\}$ and $\{b_n\}$, we receive convergent sequences

- $\{ka_n\}, k \in R, k \neq 0, \{a_n + b_n\}, \{a_n - b_n\}, \{a_n . b_n \}$.

If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, then

a) $\lim_{n\to\infty} ka_n = ka$

b) $\lim_{n\to\infty} (a_n + b_n) = a + b$

c) $\lim_{n\to\infty} (a_n - b_n) = a - b$

d) $\lim_{n\to\infty} a_n . b_n = a.b$

e) if $b_n \neq 0$ for all $n \in N$ and $b \neq 0$, then also the sequence $\{a_n/b_n\}$ is convergent and

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$$

Let $\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b$

and let for all natural numbers be defined ${a_n}^{b_n}$

Then the following limit exists

$$\lim_{n\to\infty}({a_n}^{b_n}) = (\lim_{n\to\infty} a_n)^{\lim\limits_{n\to\infty} b_n} = a^b$$

Three important limits

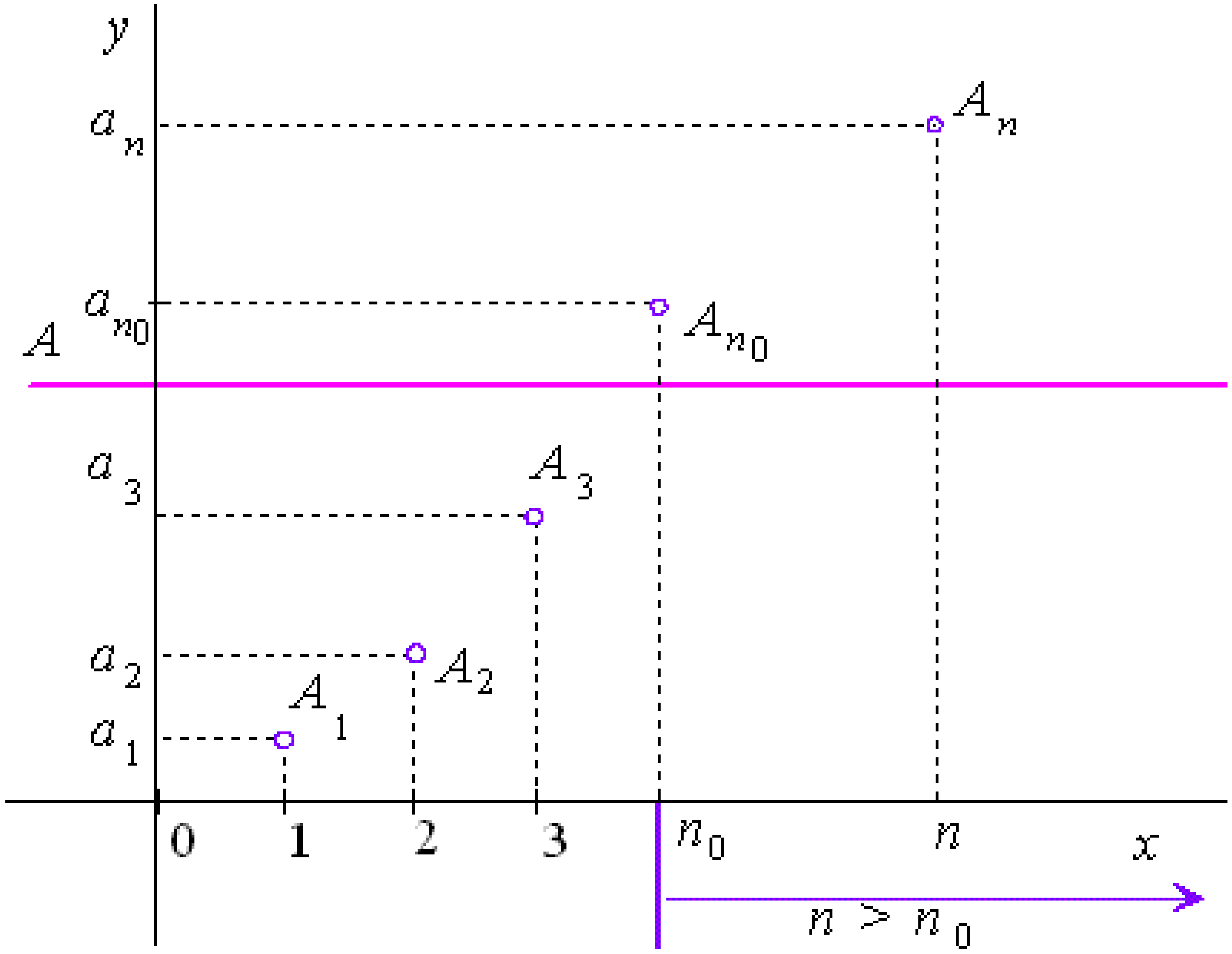
$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Euler number

$$\lim_{n\to\infty} (1+\frac{1}{n})^n = e$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Improper limit of a sequence



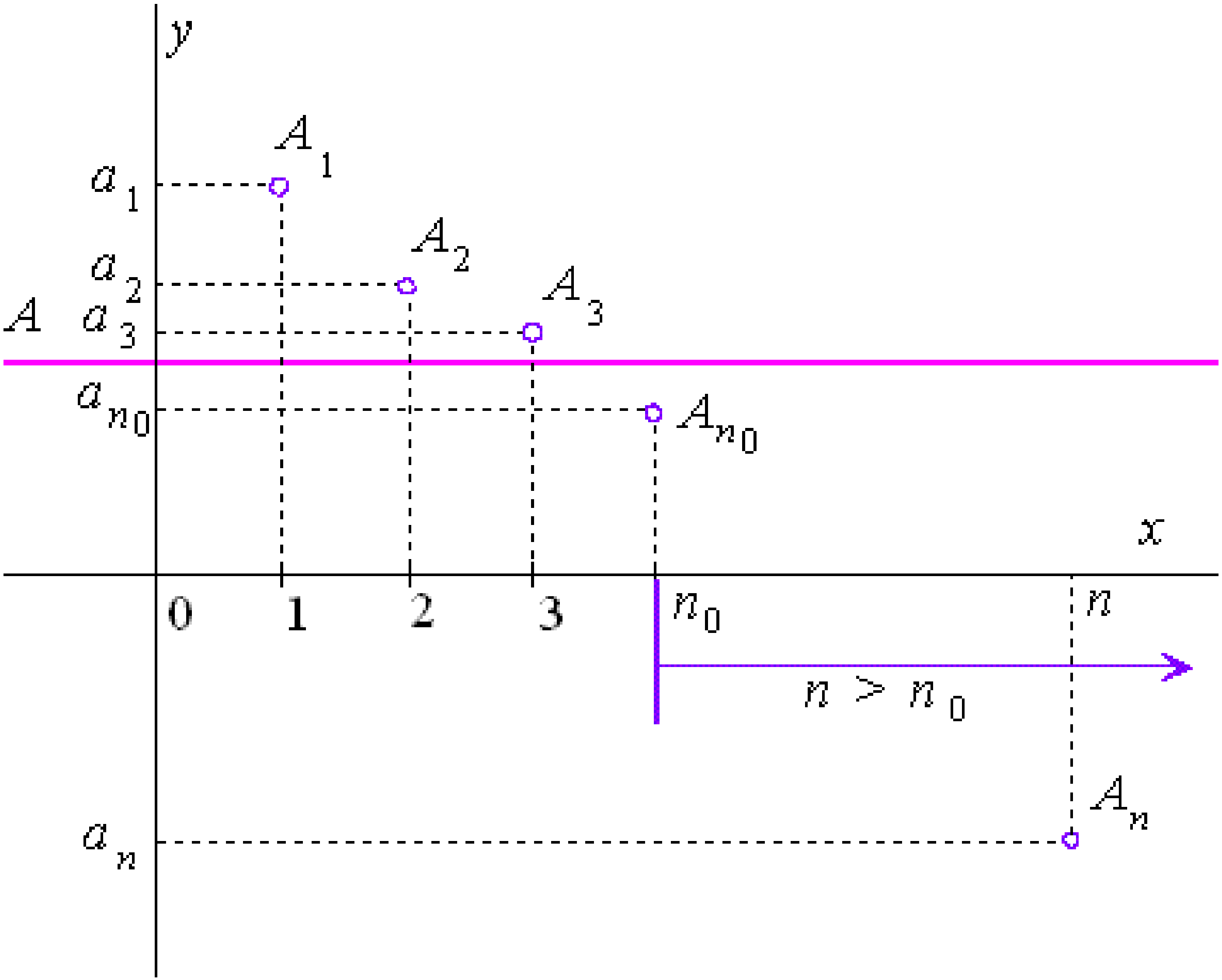
Sequence $\{a_n\}$ has an improper limit ∞, if almost all terms of the sequence $\{a_n\}$ are located in any neighbourhood $O_A(\infty)$.

This means that to any number A there exists number n_0 such, that for all $n > n_0$ holds $a_n > A$.

$$\lim_{n\to\infty} a_n = \infty$$

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \forall A > 0 \, \exists n_0 : \forall n > n_0 : a_n > A$$

Improper limit of a sequence



Sequence $\{a_n\}$ has an improper limit $-\infty$, if almost all terms of the sequence $\{a_n\}$ are located in any neighbourhood $O_A(-\infty)$.

This means that to any number A there exists number n_0 such, that for all $n > n_0$ holds $a_n < A$.

$$\lim_{n\to\infty} a_n = -\infty$$

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \forall A, \exists n_0 : \forall n > n_0 : a_n < A$$

For any sequence $\{a_n\}_{n=1}^{\infty}$

one of the following situations can be true:

- sequence has a proper limit, it is convergent

$$\lim_{n\to\infty} a_n = a$$

- sequence has an improper limit, it is divergent

$$\lim_{n\to\infty} a_n = \pm\infty$$

- there exists no proper or improper limit of the sequence, and it is called an oscillating sequence.

Rules for calculating the limits of the
sum, difference, product, quotient, powers
and roots of two sequences
are valid also for improper limits,
provided all formulas and algebraic terms,
which occur in them, are defined.

Calculations of improper limits of sequencers are performed using the following rules, $c \in R$

Sum and difference

$$\infty + \infty = \infty$$

$$-\infty - \infty = -\infty$$

$$c + \infty = \infty$$

$$c - \infty = -\infty$$

Product

$$\infty . \infty = \infty$$

$$-\infty . \infty = -\infty$$

$$c . \infty = \begin{cases} \infty, c > 0 \\ -\infty, c < 0 \end{cases}$$

$c \in R$

Quotient

$$\frac{c}{\infty} = 0$$

$$\frac{c}{-\infty} = 0$$

Power

$$\infty^{\infty} = \infty$$

$$\infty^{-\infty} = 0$$

$$\infty^{c} = \begin{cases} \infty, c > 0 \\ 0, c < 0 \end{cases}$$

$$c^{\infty} = \begin{cases} 0, c \in \langle 0,1) \\ \infty, c > 1 \end{cases}$$

Calculating limits we sometimes come to so called undetermined expressions

$$\pm\frac{\infty}{\infty}, \frac{0}{0}, \infty-\infty, 0.\infty, 0^0, 1^\infty, \infty^0$$

Special limit

Let $\lim\limits_{n\to\infty}|a_n| = \infty$ then $\lim\limits_{n\to\infty}(1+\frac{1}{a_n})^{a_n} = e$

or

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Procedure for calculation of a limit of any quotient of two polynomials in variable *n*, *P*(*n*)/*Q*(*n*) can be generalized as follows:

1) $\lim_{n\to\infty} \frac{P(n)}{Q(n)} = 0$ if the degree of *P*(*n*) is less than degree of *Q*(*n*).

2) $\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \pm\infty$ if the degree of *Q*(*n*) is less than degree of *P*(*n*) and the sign depends on coefficients at the greatest powers in *P*(*n*) and *Q*(*n*) .

3) $\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \frac{p}{q}$ if the degrees of *P*(*n*) and *Q*(*n*) are identical and *p, q* are the coefficients at the greatest power sin *P*(*n*), *Q*(*n*)).