# FUNCTIONS OF ONE REAL VARIABLE

# Concept of a function

Real function *f* of one real variable is a mapping from the set M, a subset in real numbers *R*, to the set of all real numbers *R*.

Function *f* is a rule, by which any real number *x* from set $M \subset R$ can be attached exactly one real number $y = f(x)$.

$$f : M \to R, x \mapsto y = f(x)$$

Number $x \in M$ is independent variable - argument of a function, number $y \in M$ is dependent variable – value of a function.

Set $M = D(f)$ is called domain of definition of a function, function is defined on the set *M*.

Set of all values of a function is denoted $H(f)$.

# Different definitions of function

- **Formula $y = f(x)$**

  $D(f)$ is set of all such real number, for which the formula $f(x)$ has a real meaning.

- **Table of values**

  Values of function in selected important values of variable $x$ are given explicitly

- **Word description of the relation**
- **Graph**

# Graph of a function

Set of all points [*x*, *y*] in the plane with defined Cartesian orthogonal coordinate system *Oxy*,

for which $x \in D(f)$ and $y = f(x)$

is called graph of function $f(x)$.

$$G(f) = \{[x, y] \in R^2 : x \in D(f) \wedge y = f(x)\}$$

Curve passing through all points $[x, f(x)]$.

A set of points in plane is graph of a function,

if each straight line parallel to y-axis

has at most one common point with it.

$D(f)$ is orthogonal projection of $G(f)$ onto x-axis and

$H(f)$ is orthogonal projection of $G(f)$ onto y-axis.

# Operations on functions

1. Absolute value of a function

$$f : M \to R, x \mapsto y = f(x)$$

is function $h(x)$

$$h : M \to R_0^+, x \mapsto h(x)$$

defined on $M$ and such, that for all $x \in M$ holds

$$h(x) = |f(x)|$$

# Operations on functions

2. Product of a real number *k* and function

$$f: M \to R, x \mapsto y = f(x)$$

is function *h*(*x*)

$$h: M \to R, x \mapsto h(x)$$

defined on *M* and such, that for all $x \in M$ holds

$$h(x) = k.f(x)$$

# Equality of functions

3. Functions $f(x)$ and $g(x)$ are equal if and only if they are defined on the same set of real numbers

$$D(f) = D(g)$$

and for any $x$ from this domain of definition holds

$$f(x) = h(x)$$

## Sum, difference and product of two functions

4. Let two real functions be given

- $f(x)$ defined on the set $M_1 \subset R$
- $g(x)$ defined on the set $M_2 \subset R$.

Function $h(x)$, with the domain of definition

$M = M_1 \cap M_2$ is called

- sum of functions, if for all $x \in M$ holds

$$h(x) = f(x) + g(x)$$

- difference of functions, if for all $x \in M$ holds

$$h(x) = f(x) - g(x)$$

- product of functions, if for all $x \in M$ holds

$$h(x) = f(x).g(x)$$

# Quotient of two functions

5. Let two functions be given

$f(x)$ defined on the set $M_1 \subset R$

$g(x)$ defined on the set $M_2 \subset R$.

Let $M \subset M_1 \cap M_2$ be the set of all such real numbers, for which the inequality holds: $g(x) \neq 0$.

Function $h(x)$, with the domain of definition $M$ is called

- quotient of functions, if for all $x \in M$ holds

$$h(x) = \frac{f(x)}{g(x)}$$

# Composite functions

Function $h(x) = (f \,.\, g)(x)$ is called a composite function composed from functions $f(x)$ and $g(x)$ if and only if domain of definition of function $h(x)$ is set of all such numbers from the domain of definition $D(g)$ of function $g(x)$, in which the value of function $g(x)$ is a number from the domain of definition $D(f)$ of function $f(x)$ and for any number $x \in M$ holds

$$h(x) = f\,[g(x)]$$

Value of function $h(x)$ in number $x$ equals to the value of function $f(u)$ in number $u = g(x)$. Function $f(u)$ is the major (outside) component, function $u = g(x)$ is the minoir (inside) component of the composite function $h(x)$.

# One-to-one function

Function $f(x)$ defined on the set $D(f)$ is denoted as one-to-one function, if for any two numbers $x_1$, $x_2$ from $D(f)$ holds:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

A function $f$ is one-to-one if each straight line parallel to $x$-axis has at most one common point with $G(f)$.

# Inverse function

Let function $f(x)$ be defined on the set $D(f)$ and its set of values be $H(f)$.
Function $f^{-1}(x)$ defined on the set $H(f)$, with values in the set $D(f)$ is called inverse function to the function $f(x)$,
if for any number $b$ from $H(f)$ holds:

$$f^{-1}(b) = a \Leftrightarrow f(a) = b$$

Graphs of inverse functions $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the line $x = y$.

# Bounded function

Function $f(x)$ defined on the set $D(f\,)$ is called bounded (bounded above, bounded below), iff there exists a real number $K$ such, that for all $x$ from $D(f\,)$ holds:

$$\left|f(x)\right| \leq K \quad (f(x) \leq K,\ f(x) \geq K)$$

It means, that a function is bounded (bounded above, bounded below), if its range $H(f)$ is a bounded (bounded above, bounded below) set of real numbers.

Function that is not bounded is called unbounded function.

# Monotone functions

Function $f(x)$ defined on the set $M$ is called increasing, if

$$\forall x_1, x_2 \in M, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

decreasing, if

$$\forall x_1, x_2 \in M, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

non-decreasing, if

non-increasing, if $\forall x_1, x_2 \in M, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

$$\forall x_1, x_2 \in M, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

The above functions are said to be monotone on $M$, incresing and decresing functions are said to be **strictly monotone**.

# Even and odd functions

Let function $f(x)$ be defined on the set $M$, which contains with any number $x$ also number $-x$.

Function $f(x)$ is said to be even on $M$, if

$$\forall x \in M \Rightarrow f(-x) = f(x)$$

Function $f(x)$ is said to be odd on $M$, if

$$\forall x \in M \Rightarrow f(-x) = -f(x)$$

Graph of an even function is symmetric with respect to the y-axis, while graph of an odd function is symmetric with respect to the origin $O$ of the coordinate system.

# Periodic function

Let function $f(x)$ be defined on the set $M$
and $p$ be a positive real number.

Function $f$ is called periodic with the period $p$, if

1. for any $p \in M$ also number $x \pm p \in M$

2. for all $x \in M$ holds

$$f(x \pm p) = f(x)$$