Determinants

Determinant of a matrix

Determinant of matrix $\boldsymbol{A}$ is number denoted det $\boldsymbol{A} = |\boldsymbol{A}|$, which can be calculated in the following way:

1. If $\boldsymbol{A}$ is matrix of rank $n = 1$, i.e. $\boldsymbol{A} = (a_{11})$, then

   det $\boldsymbol{A} = |\boldsymbol{A}| = a_{11}$

2. For any $n \geqq 2$ determinant is number

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{11}|\boldsymbol{A}_{11}| - a_{12}|\boldsymbol{A}_{12}| + a_{13}|\boldsymbol{A}_{13}| - \dots + (-1)^{1+n}|\boldsymbol{A}_{1n}|$$

where $|\boldsymbol{A}_{1j}|$, for $j = 1, 2, \dots, n$ are sub-determinants of matrix obtained from matrix $\boldsymbol{A}$ by deleting its first row and j-th column.

Determinant of matrix *A* of rank 2

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of matrix $\boldsymbol{A}$ of rank 3

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Sarrus' rule or **Sarrus' scheme**

method and a memorization scheme to compute the determinant of a 3×3 matrix

named after the French mathematician Pierre Frédéric Sarrus.

Sarus rule

- Write down the first two rows of the matrix as the fourth and the fifth row
- Add the products of the diagonals going from top left to bottom right
- and subtract the products of the diagonals going from top right to bottom left

$$\begin{array}{c} + \\ + \\ + \\ - \\ - \\ - \end{array}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

Write out the first 2 columns of the matrix to the right of the 3rd column, so that you have 5 columns in a row. Then add the products of the diagonals going from top to bottom (solid) and subtract the products of the diagonals going from bottom to top (dashed)

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

- Determinant of a matrix with at least one zero row or column (two equal rows or columns, or with linearly dependent rows and columns), equals zero.

- Determinant of a regular matrix (with none zero rows or columns, or such that no row is a linear combination of other rows) is nonzero.

- Determinant of matrix $\boldsymbol{A}$ and its transposed matrix $\boldsymbol{A}^T$ are equal.

Cramer rule

System of 2 equations with 2 unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Denoting

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

If $D \neq 0$, then system has a unique solution

$$(x_1, x_2) = \left(\frac{D_1}{D}, \frac{D_2}{D} \right)$$

Cramer rule

System of three equations with three unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Denoting

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

If $D \neq 0$, then system has a unique solution

$$(x_1, x_2, x_3) = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D} \right)$$