DIFFERENTIAL EQUATIONS
of the SECOND ORDER
with SPECIAL RIGHT-HAND TERMS

Steps to solve non-homogeneous LDE of order 2

$$y'' + p_1 y' + p_2 y = q(x)$$

1. Find general solution of homogeneous LDE

$y'' + p_1 y' + p_2 y = 0$

$$y_h = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in R$$

2. Find one solution $\tilde{y}$ of non-homogeneous LDE

3. General solution of non-homogeneous LDE is

$$y_v = y_h + \tilde{y} = c_1 y_1 + c_2 y_2 + \tilde{y}, \quad c_1, c_2 \in R$$

Method of variation of constants – to find particular solution $\tilde{y}$ of non-homogeneous LDE

$$y'' + p_1 y' + p_2 y = q(x)$$

$$y'' + p_1 y' + p_2 y = 0 \Rightarrow y_h = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in R$$

$$\tilde{y} = c_1(x) y_1 + c_2(x) y_2$$

$$c_1(x) = \int \frac{W_1}{W} dx \quad c_2(x) = \int \frac{W_2}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad W_1 = \begin{vmatrix} 0 & y_2 \\ q(x) & y_2' \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & q(x) \end{vmatrix}$$

Solution of non-homogeneous LDE of order 2 with constant coefficients and special right-hand term

a) let the right-hand term be in the form

$$q(x) = e^{\alpha x} P_n(x)$$

where $\alpha \in R$ and $P_n(x)$ be a polynomial of degree n, than solution is

$$\tilde{y} = x^k e^{\alpha x} P_n^*(x)$$

Number $k \in \{0, 1, 2\}$ determines multiplicity of the root α of the respective characteristic equation

and $P^*_n(x)$ is unknown polynomial of degree n, whose coefficients can be determined by method of indefinite coefficients.

Solution of non-homogeneous LDE of order 2 with constant coefficients and special right-hand terms

b) let the right-hand term be in the form

$$q(x) = e^{\alpha x}(P_n(x)\cos \beta x + Q_m(x)\sin \beta x)$$

where $\alpha, \beta \in R$ and $P_n(x)$, $Q_m(x)$ be polynomials of orders n and m, than solution is

$$\tilde{y} = x^k e^{\alpha x}(P_s^*(x)\cos \beta x + Q_s^*(x)\sin \beta x)$$

Number $k \in \{0, 1\}$ determines multiplicity of the complex number $\alpha + \mathrm{i}\beta$, which is the root of the respective characteristic equation and

$P^*_s(x)$, $Q^*_s(x)$ are unknown polynomials of degree $s = \max\{m, n\}$.