

DIFFERENTIAL EQUATIONS of the SECOND ORDER

Linear differential equations of order two with constant coefficients

is a differential equation in the form

$$y'' + p_1 y' + p_2 y = q(x)$$

where p_1, p_2 are real numbers and $q(x)$ is a continuous function on interval (a, b) .

If $q(x) = 0, \forall x \in (a, b)$, equation is called homogeneous, it is called non-homogeneous otherwise.

There exists exactly one solution of LDE

$$y'' + p_1 y' + p_2 y = q(x)$$

satisfying initial conditions

$$y(x_0) = y_0, y'(x_0) = y_1, x_0 \in (a, b).$$

HOMOGENEOUS LDE of order 2

Two solutions y_1, y_2 of equation in the form

$$y'' + p_1 y' + p_2 y = 0$$

are linearly dependent on R , if there exists such number k , that for all $x \in R$ holds $y_1 = k y_2$.

If two solutions y_1, y_2 of homogeneous LDE are not linearly dependent on R , they are said to be **linearly independent** on R .

Function $y_1 = 0$ is solution of any homogeneous LDE, and it is called a trivial solution.

Let functions y_1, y_2 be defined on interval J and let them be differential on this interval, while their derivatives are y_1', y_2'

Determinant
$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is called Wronski determinant of functions y_1, y_2 and it is denoted $W(y_1, y_2)$.

Solutions y_1, y_2 of homogeneous LDE are linearly independent on J if and only if

$$W(y_1, y_2) \neq 0, \forall x \in J.$$

System of two linearly independent solutions of a homogeneous LDE is called **fundamental system of solutions** of this differential equation.

Let functions y_1, y_2 form a fundamental system of solutions of homogeneous LDE

$$y'' + p_1 y' + p_2 y = 0$$

then a general solution is in the form

$$y_h = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in R$$

General solution of homogeneous LDE of order 2
with constant coefficients

$$y'' + p_1 y' + p_2 y = 0$$

Quadratic equation in the form

$$r^2 + p_1 r + p_2 = 0$$

is called characteristic equation of the respective
LDE, with roots r_1, r_2 .

1. Characteristic equation has 2 real roots

Functions y_1, y_2 form a fundamental system of solutions

$$r_1 \rightarrow y_1 = e^{r_1 x}$$

$$r_2 \rightarrow y_2 = e^{r_2 x}$$

while general solution is in the form

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad c_1, c_2 \in R, x \in R$$

2. Characteristic solution has one double root

$$r = r_1 = r_2$$

$$r \rightarrow y_1 = e^{rx}$$

$$y_2 = xe^{rx}$$

Functions y_1, y_2 form a fundamental system of solutions and general solution is in the form

$$y_h = c_1 e^{rx} + c_2 x e^{rx}, \quad c_1, c_2 \in R, x \in R$$

3. Characteristic equation has 2 complex conjugate roots

$$r_1 = a + ib, r_2 = a - ib$$

$$r_1 = a + ib \rightarrow y_1 = e^{ax} \cos bx \quad r_2 = a - bi \rightarrow \emptyset$$

$$y_2 = e^{ax} \sin bx$$

Functions y_1, y_2 form a fundamental system of solutions and general solution is in the form

$$y_h = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx, \quad c_1, c_2 \in R, x \in R$$