Matrices
and operations on matrices

Matrices

Let m, n be natural numbers.
A rectangular array of $m.n$ real numbers a_{ik}, where $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$, arranged to m rows and n columns, enclosed in parenthesis,
is called a matrix of the type m x n and written in the way

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ik})$$

The set of numbers $(a_{i1}, a_{i2}, \ldots, a_{in})$ is called the i-th row of matrix.
These are such elements of matrix that have the same first indeces, and that are written in one, namely i-th row of the array.
The set of numbers $(a_{1k}, a_{2k}, , \ldots, a_{mk})$ is called the k-th column of the matrix.
These are such elements of matrix that have the same second index, and that are written in one, namely k-th column of the array.

Each row or column of the matrix can be regarded as a vector.

Matrix

Matrix with m rows and n columns is called the matrix of type m x n. Matrix has a rectangular shape for $m \neq n$ therefore we speak about rectangular matrix.

Matrix of type n x n, which means number of matrix rows equals to the number of its columns, is called the square matrix of type n x n.

Elements of a square matrix, whose row and column indeces are equal, therefore they are on one from 2 diagonals of this square array, are called diagonal elements.
Vector (a_{11}, a_{22}, ..., a_{nn}) is the major (principle) diagonal of the matrix, vector (a_{n1}, ..., a_{1n}) is the minor diagonal of the matrix.

Square matrix with all zero elements except the elements of the major diagonal is called the diagonal matrix.

Diagonal matrix of degree n, with all elements on the major diagonal equal to number 1, is called the unit matrix (of rank n) denoted $\boldsymbol{E}$.

Matrix

Matrix of type m x n, whose all elements are equal to zero, is called null matrix and it is denoted as $\mathbf{0}$.

Let $\boldsymbol{A} = (a_{ik})$ is the matrix of type m x n. Matrix $\boldsymbol{A} = (a_{ki})$ of type n x m, whose elements satisfy the following equalities

$$a_{ki} = a_{ik}, \; i = 1, 2, ..., m, \quad k = 1, 2, ..., n$$

is called the transposed matrix to the matrix $\boldsymbol{A}$, and it is denoted as $\boldsymbol{A}^T$.

Matrix $\boldsymbol{A}^T$ is generated from the matrix $\boldsymbol{A}$ by interchanging its rows and columns, it means rows of the matrix $\boldsymbol{A}^T$ are columns of the matrix $\boldsymbol{A}$ and columns of the matrix $\boldsymbol{A}^T$ are rows of the matrix $\boldsymbol{A}$.

In case of a square matrix $\boldsymbol{A}$ its transposed matrix $\boldsymbol{A}^T$ is also a square matrix generated from $\boldsymbol{A}$ by transposition around the major diagonal.

Diagonal matrix of type n x n is equal to its transposed matrix.

Operations on matrices

1. Equality of matrices

Matrices $\boldsymbol{A} = (a_{ik})$ and $\boldsymbol{B} = (b_{ik})$ are equal, $\boldsymbol{A} = \boldsymbol{B}$,
iff they are of the same type m x n and

$$a_{ik} = b_{ik}\ ,\ i = 1, 2, ..., m,\quad k = 1, 2, ..., n$$

2. Matrix addition

Let $\boldsymbol{A} = (a_{ik})$ and $\boldsymbol{B} = (b_{ik})$ are matrices of the same type m x n.
Matrix $\boldsymbol{C} = (c_{ik})$ of type m x n, whose elements c_{ik} are

$$c_{ik} = a_{ik} + b_{ik}\ ,\ i = 1, 2, ..., m,\quad k = 1, 2, ..., n$$

is called sum of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ written as $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$.

Sum is not defined for matrices, which are not of the same type.
$\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$ for arbitrary matrix $\boldsymbol{A}$ and null matrix $\boldsymbol{0}$, both of type m x n.
Sum of two diagonal matrices is also a diagonal matrix.

Operations on matrices

3. Multiplication of a matrix by a number

Let $\boldsymbol{A} = (a_{ik})$ be a matrix of type m x n and let c be a number.
Matrix $\boldsymbol{B} = (b_{ik})$ of type m x n, for whose elements b_{ik} holds

$$b_{ik} = c \,.\, a_{ik} \,,\; i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n$$

is called product of number c and matrix $\boldsymbol{A}$ written as $\boldsymbol{B} = c.\boldsymbol{A}$.

4. Multiplication of matrices

Let $\boldsymbol{A} = (a_{ik})$ be a matrix of type m x p and $\boldsymbol{B} = (b_{ik})$ be a matrix of type p x n.
Matrix $\boldsymbol{C} = (c_{ik})$ of type m x n, whose elements c_{ik} satisfy

$$c_{ik} = \sum_{j=1}^{p} a_{ij} \,.\, b_{jk} \,,\; i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n$$

is called product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (in the given order)
and written as $\boldsymbol{C} = \boldsymbol{A} \,.\, \boldsymbol{B}$.
Multiplication of matrices, in general, is not commutative, $\boldsymbol{A} \,.\, \boldsymbol{B} \neq \boldsymbol{B} \,.\, \boldsymbol{A}$.

Product is not defined for such two matrices, whose number of columns of the first matrix is not equal to the number of rows of the second matrix.

Operations on matrices

Two rows of a matrix are called linearly independent, if none of them is the multiple of the other one.
Rows of a matrix are linearly indepenedent, if none of them is a linear combination of the other rows of ther matrix.

Matrix is called a regular matrix, if it does not contain any linearly dependent (and therefore also zero) rows.

Number of linearly independent rows is rank of matrix.

For a square matrix $\boldsymbol{A}$ of rank n and a unit matrix $\boldsymbol{E}$ of the same rank n, matrix $\boldsymbol{A}^{-1}$, for which

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{E}$$

is called the inverse matrix of the matrix $\boldsymbol{A}$.

For any regular square matrix $\boldsymbol{A}$ there exists an inverse matrix $\boldsymbol{A}^{-1}$.

Operations on matrices

For any square matrix $\boldsymbol{A}$ of rank n
and a unit matrix $\boldsymbol{E}$ of the same rank n, holds

$$\boldsymbol{A} \, . \, \boldsymbol{E} = \boldsymbol{E} \, . \, \boldsymbol{A} = \boldsymbol{A}$$

Unit matrix $\boldsymbol{E}$ plays in multiplication of matrices the same role as number 1 in multiplication of numbers.

Product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is not commutative,
nor in the case of square matrices, $\boldsymbol{A} \, . \, \boldsymbol{B} \neq \boldsymbol{B} \, . \, \boldsymbol{A}$.

Product of two diagonal matrices is again a diagonal matrix.

Product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be a zero matrix even if both matrices are non-zero.

Rules for operations on matrices

1. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
2. $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$
3. $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$
4. $c(\boldsymbol{A} + \boldsymbol{B}) = c\boldsymbol{A} + c\boldsymbol{B}$
5. $(c + d)\boldsymbol{A} = c\boldsymbol{A} + d\boldsymbol{A}$
6. $c(\boldsymbol{A} \cdot \boldsymbol{B}) = (c\boldsymbol{A}) \cdot \boldsymbol{B} = \boldsymbol{A} \cdot (c\boldsymbol{B})$
7. $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$
8. $(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$
9. $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$
10. $\boldsymbol{A} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot \boldsymbol{A} = \boldsymbol{A}$
11. $\boldsymbol{0} \cdot \boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{0} = \boldsymbol{0}$