IMPROPER INTEGRALS

Improper integral on unbounded interval

Let function $f(x)$ be defined on unbounded interval $\langle a, \infty)$ and integrable on interval $\langle a, b\rangle$ for all $b > a$.

If there exists a proper limit

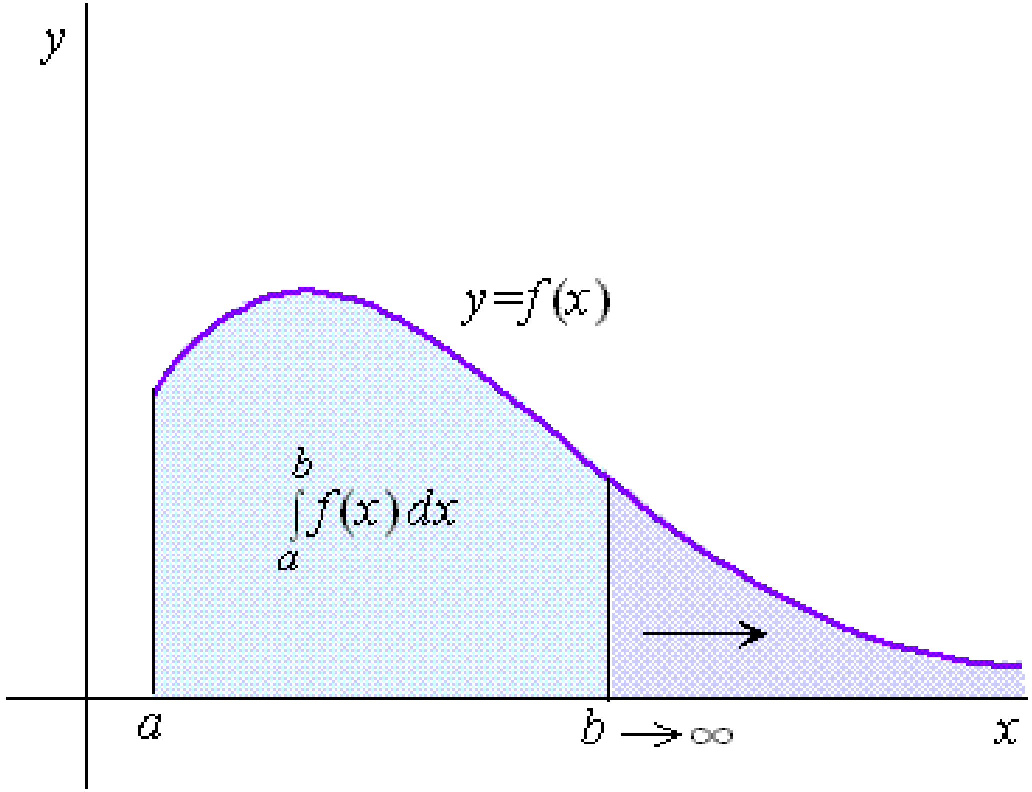
$$\lim_{b\to\infty}\int_a^b f(x)dx = \int_a^\infty f(x)dx$$

then it is called the improper integral of function $f(x)$ on interval $\langle a, \infty)$, and improper integral is said to be converging.

If the proper limit does not exist, the improper integral is said to be diverging.

Let function $f(x)$ be continuous and non-negative on interval $\langle a, \infty)$, then for any $b > a$ the improper integral

$$\lim_{b\to\infty}\int_a^b f(x)dx = \int_a^\infty f(x)dx$$



determines the area of a curvilinear trapezoid.

If the improper integral is converging for $b \to \infty$, its value is the area of a planar region unbounded from right, bounded by line $x = a$ from left, by axis x below and by graph of function $f(x)$ above.

Improper integral on unbounded intervale

Let function $f(x)$ be defined on interval $(-\infty, b\rangle$ and integrable on interval $\langle a, b\rangle$ for all $a < b$.

If there exists a proper limit

$$\lim_{a \to -\infty} \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx$$

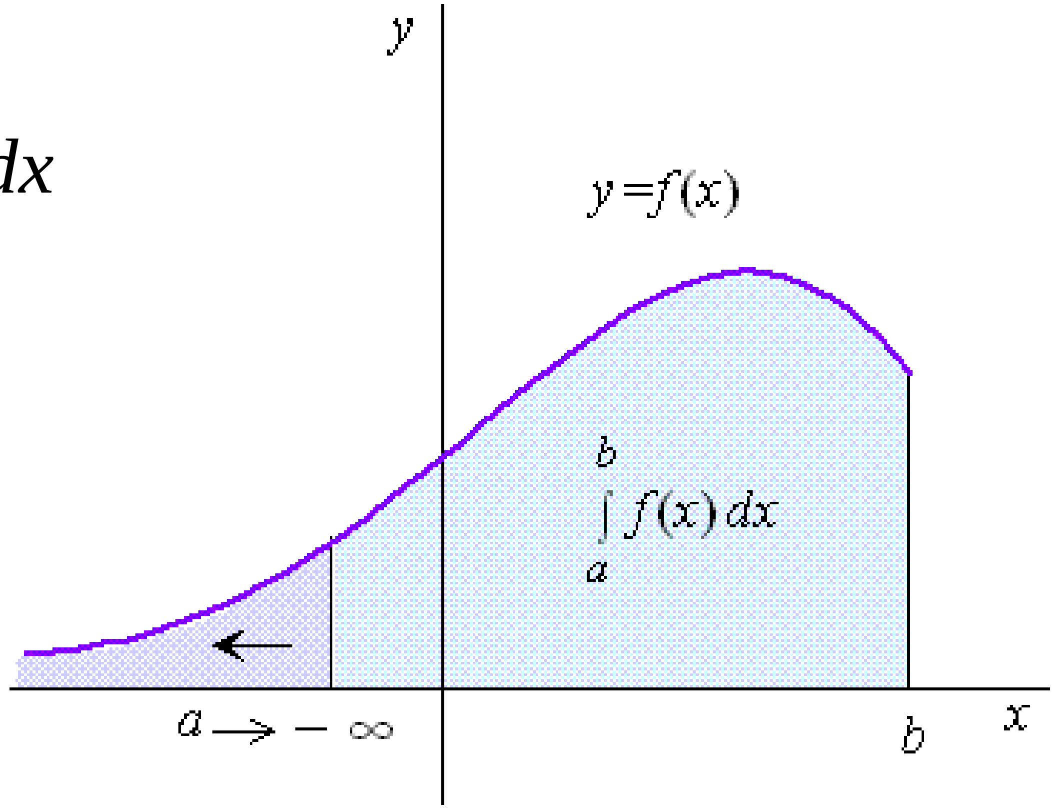
then it is called the improper integral of function $f(x)$ on intervale $(-\infty, b\rangle$, and improper integral is said to be converging.

If the proper limit does not exist, the improper integral is said to be diverging.

Let function $f(x)$ be continuous and non-negative on interval $(-\infty, b\rangle$, then for any $a < b$ the improper integral

$$\lim_{a \to -\infty} \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx$$

determines the area of a curvilinear trapezoid.



If the improper integral is converging for $a \to -\infty$, its value is the area of a planar region unbounded from left, bounded by line $x = b$ from right, by axis x below and by graph of function $f(x)$ above.

Improper integral on unbounded interval

Let function *f*(*x*) be defined on interval (- ∞, ∞) and let *c* be any real number.

If the following improper integrals exist

$$\int_{-\infty}^{c} f(x)dx \qquad \int_{c}^{\infty} f(x)dx$$

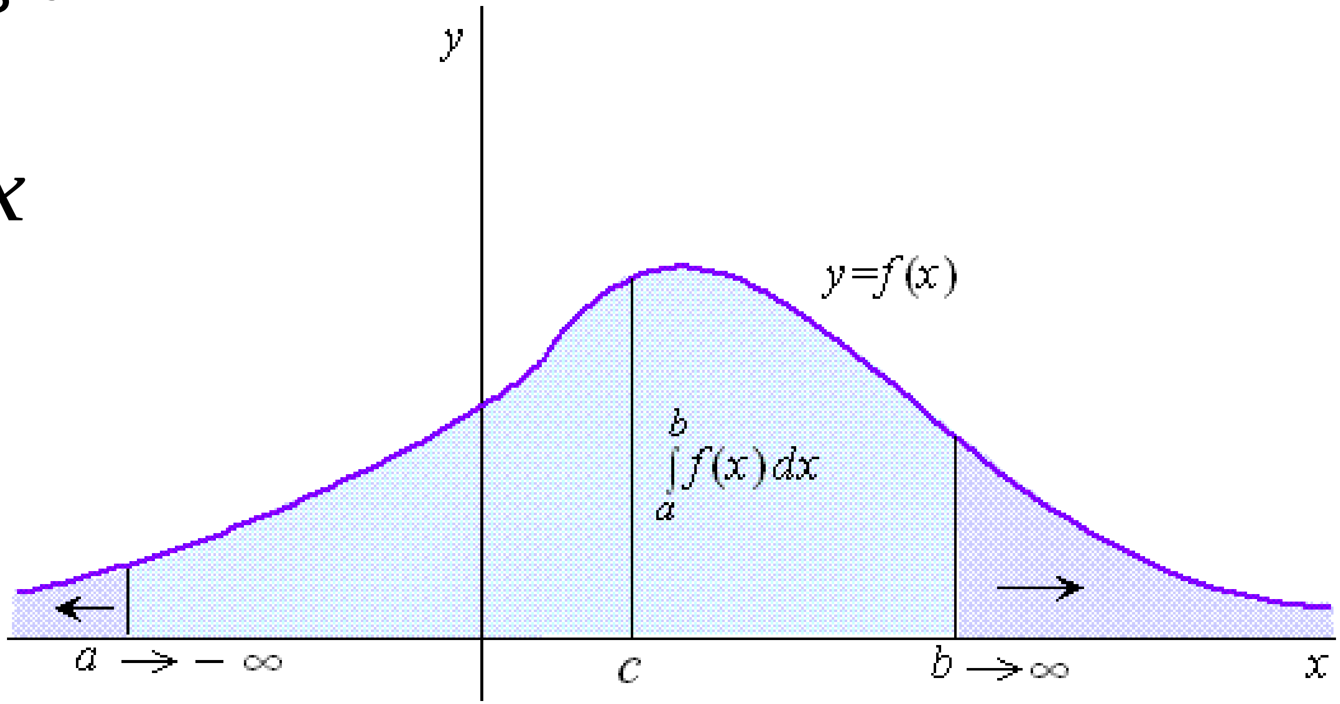
then their sum is denoted as improper integral of function *f*(*x*) on interval (-∞, ∞).

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx, c \in R$$

If at least one from improper integrals $\int_{-\infty}^{c} f(x)dx, \int_{c}^{\infty} f(x)dx$ be diverging, then also improper integral

$$\int_{-\infty}^{\infty} f(x)dx$$

is diverging.



Let function $f(x)$ be continuous and non-negative on interval (- ∞, ∞), and let improper integral be converging. Value of the integral is the area of a planar region unbounded form left and right, and bounded by axis x below and by graph of function $f(x)$ above.

Improper integral from unbounded function

Let function $f(x)$ be defined on interval $\langle a, b)$ and let it be unbounded on some neighbourhood of point b

$$\lim_{x \to b^-} f(x) = \pm\infty$$

If it is integrable on every interval $\langle a, b - \varepsilon\rangle$, $\varepsilon > 0$, $b - \varepsilon > a$ and the proper limit exists

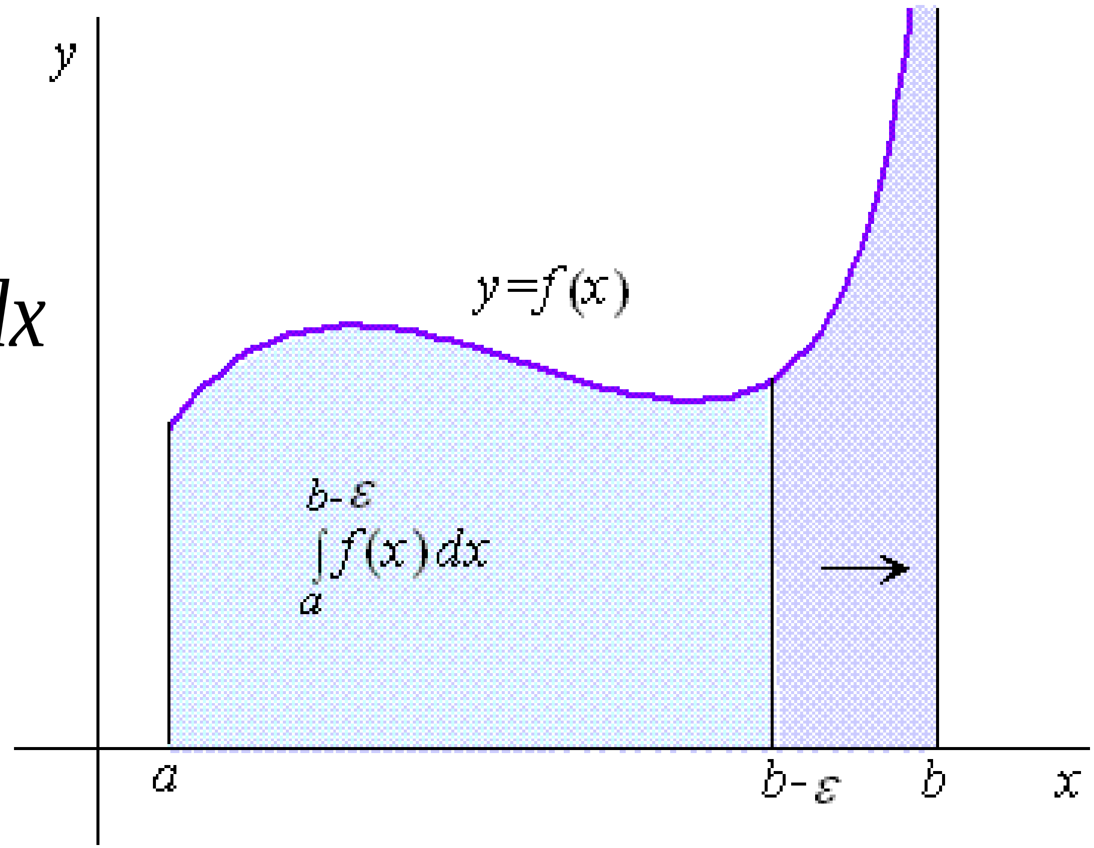
$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)dx$$

then this limit is called improper integral from unbounded function $f(x)$ on interval $\langle a, b\rangle$

$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)dx = \int_a^b f(x)dx$$

Let function $f(x)$ be continuous and non-negative on interval $\langle a, b)$, then the improper integral

$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)dx = \int_a^b f(x)dx$$



determines the area of the unbounded planar region from right, bounded by line $x = a$ from left, by axis x below and by graph of function $f(x)$ above.

Improper integral from unbounded function

Let function $f(x)$ be defined on interval $(a, b\rangle$ and let it be unbounded on some neighbourhood of point a

$$\lim_{x \to a^+} f(x) = \pm\infty$$

If it is integrable on every interval $\langle a + \varepsilon, b\rangle$, $\varepsilon > 0$, $a + \varepsilon < b$ and there exists proper limit

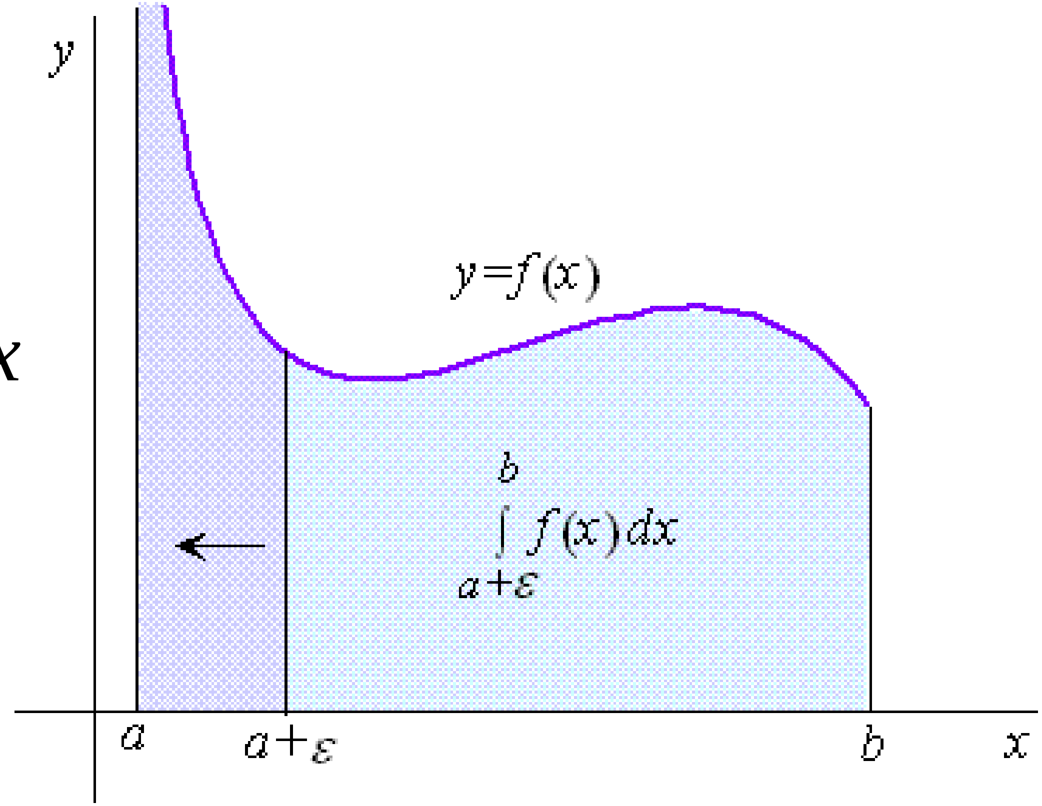
$$\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)dx$$

then this limit is called improper integral from unbounded function $f(x)$ on interval $\langle a, b\rangle$

$$\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)dx = \int_{a}^{b} f(x)dx$$

Let function $f(x)$ be continuous and non-negative on interval $(a, b\rangle$, then improper integral

$$\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)dx = \int_{a}^{b} f(x)dx$$



determines the area of the unbounded planar region from right, bounded by line $x = b$ from right, by axis x below and by graph of function $f(x)$ above.

Improper integral from unbounded function

Let function *f*(*x*) be defined on interval (*a*, *b*) and let it be unbounded on some neighbourhood of point *a* and on some neighbourhood of point *b*.

Let *c* be any number from interval (*a*, *b*).

If there exist improper integrals

$$\int_a^c f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^c f(x)dx, \quad \int_c^b f(x)dx = \lim_{\varepsilon \to 0} \int_c^{b-\varepsilon} f(x)dx$$

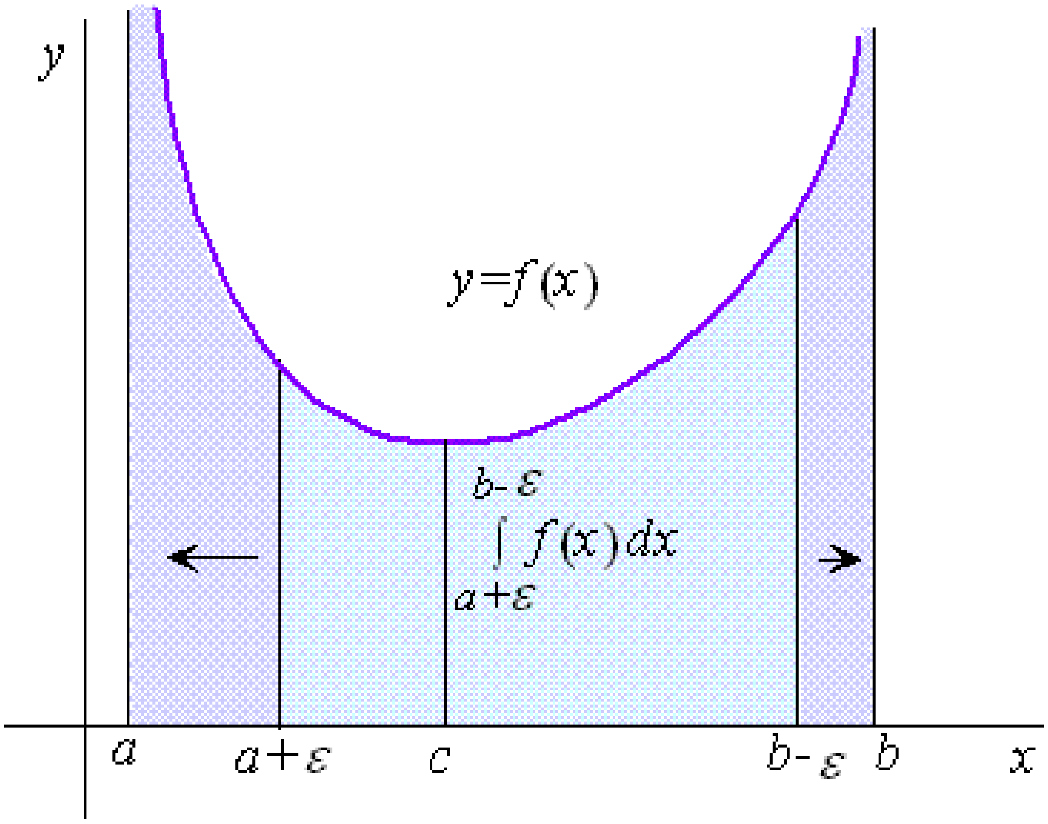
then their sum is called improper integral of function *f*(*x*) on interval $\langle a, b \rangle$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let function *f*(*x*) be continuous and non-negative on interval (*a*, *b*), then improper integral

$$\int_a^c f(x)dx =$$

$$= \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b-\varepsilon} f(x)dx$$



determined the area of an unbounded planar region, bounded by line $x = a$ from left, by line $x = b$ from right, by axis *x* below and by graph of function *f*(*x*) above.

Improper integral from unbounded function

Let function $f(x)$ be not bounded on certain neighbourhood of point $c \in (a, b)$ and let it be integrable on all intervals $\langle a, c-\varepsilon \rangle$ and $\langle c+\varepsilon, b \rangle$, for $\varepsilon > 0$, $c-\varepsilon > a$, $c+\varepsilon < b$.

If both following integrals exist

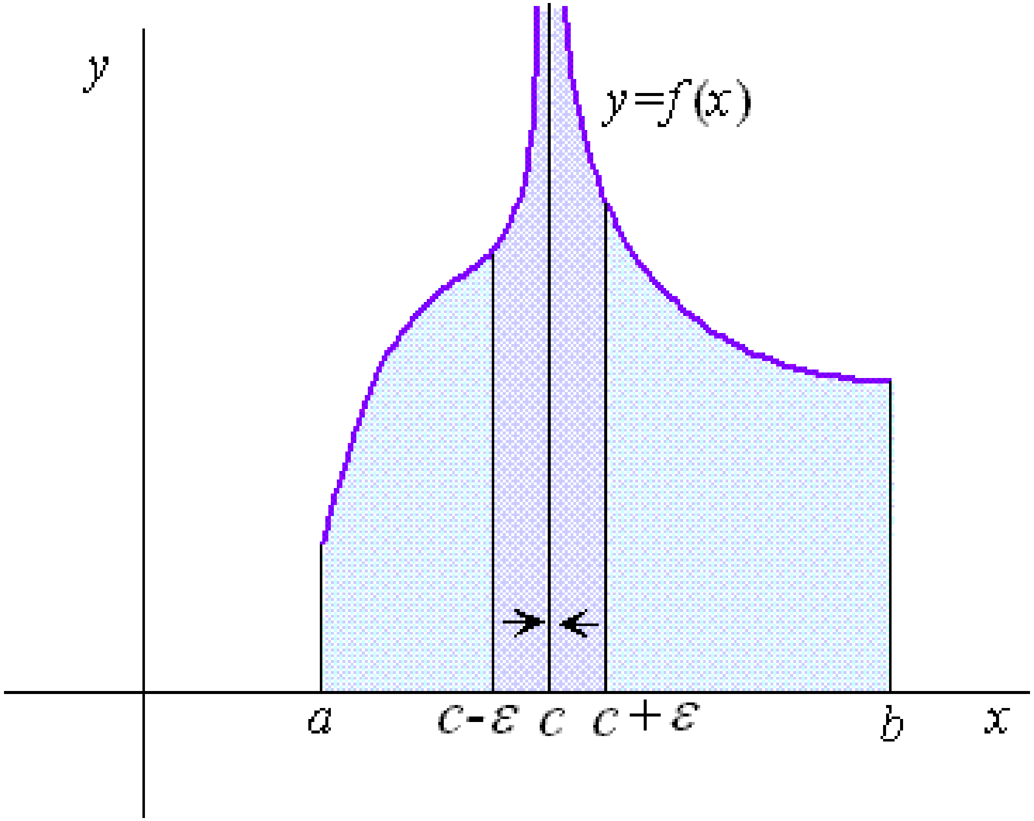
$$\int_a^c f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx, \quad \int_c^b f(x)dx = \lim_{\varepsilon \to 0} \int_{c+\varepsilon}^b f(x)dx$$

then their sum is called the improper integral of function $f(x)$ on interval $\langle a, b \rangle$ and denote

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let function *f*(*x*) be continuous and non-negative on interval $\langle a, b\rangle$ - up to one point if discontinuity $c \in (a, b)$, then improper integral

$$\int_a^b f(x)dx$$



determines the area of unbounded planar region, which is bounded by line $x = a$ from left, by line $x = b$ from left, by axis x below and by graph of function $f(x)$ above.

IMPROPER INTEGRALS

I. Integrals on unbounded intervals

a) $\langle a, \infty)$

b) $(-\infty, b\rangle$

c) $(-\infty, \infty)$

II. Integrals of unbounded functions

a) at point b

b) at point a

c) at points a and b

d) at point $c \in (a, b)$