

INTEGRATION of SPECIAL TYPES of FUNCTIONS

Integration of some rational functions

$$1. \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c, a > 0$$

$$2. \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + c, a > 0$$

$$3. \int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln |x^2 \pm a^2| + c, a > 0$$

$$4. \int \frac{ax + b}{x^2 + px + q} dx$$

a) complete the denominator to the square

b) use substitution

$$t = x + \frac{p}{2}$$

c) by some algebraic manipulations we can receive one from the basic types of integrals 1.- 3.

or integral of type

$$\int \frac{1}{x^2} dx$$

$$5. \int \frac{P(x)}{x^2 + px + q} dx$$

where the polynomial $P(x)$ degree is at least 2.

Deviding the numerator by denominator receive polynomial $Q(x)$ and a rational function, which can be integrated as in 4.

$$\frac{P(x)}{x^2 + px + q} = Q(x) + \frac{ax + b}{x^2 + px + q}$$

Integration of some irrational functions

1. If the following expression appears in the integrated function

$$\sqrt[n]{ax + b}, a \neq 0$$

use substitution

$$t = \sqrt[n]{ax + b}$$

$$2. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + c, a > 0$$

$$3. \int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + c,$$

$$4. \int \frac{1}{\sqrt{Ax^2 + Bx + C}} dx, A \neq 0$$

Complete denominator to square and then by algebraic manipulations receive integral of one from types 2.- 3.

Integration of some goniometric functions

1. $\int f(\sin x) \cos x dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int f(t) dt$
2. $\int f(\cos x) \sin x dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x dx \end{array} \right| = -\int f(t) dt$
3. $\int \sin^2 x dx \quad \sin^2 x = \frac{1 - \cos 2x}{2}$
4. $\int \cos^2 x dx \quad \cos^2 x = \frac{1 + \cos 2x}{2}$

Integration of some goniometric functions

$$5. \int f(\cos x, \sin x) dx \quad t = \tan \frac{x}{2}, 2 \arctan t = x$$

$$\frac{2dt}{1+t^2} = dx$$

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

Some integrals cannot be expressed by means of elementary functions, these are called elliptic (or hyper-elliptic) integrals.

$$\int e^{-x^2} dx, \int \frac{e^x}{x} dx, \int \frac{1}{\ln x} dx$$

$$\int \frac{\sin x}{x} dx, \int \frac{\cos x}{x} dx$$