MONOTONICITY
FUNCTION EXTREMA

Let function f be continuous on interval $J \subset R$ and let it be differentiable at all inner points of this interval, i.e. there exists derivative f'.

Function f is increasing (decreasing) on interval J iff, for all x from the interior of interval J holds

$$f'(x) \geq 0 \quad (f'(x) \leq 0),$$

while equality can hold only in those points, which do not form interval $J_1 \subset J$.

f is said to be non-increasing on $J \subset R$ if

$$f'(x) \leq 0, \forall x \in J$$

f is said to be non-decreasing on $J \subset R$ if

$$f'(x) \geq 0, \forall x \in J$$

In geometric terms:

Differentiable functions increase on intervals where their graphs have positive slopes and decrease on intervals where their graphs have negative slopes.

Local extrema

Let function $f(x)$ be defined at point x_0.

Function value $f(x_0)$ is called local maximum (minimum) of function f, if there exists such neighbourhood

$O(x_0) \subset D(f)$ of point x_0, that for all $x \in O(x_0)$ holds

$f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$).

If for $\forall x \in O(x_0)$, $x \neq x_0$ holds $f(x) < f(x_0)$ ($f(x) > f(x_0)$), we speak about strict local maximum (minimum) of function.

Local maximum and local minimum are commonly called local extrema of function.

Point, at which function achieves local extremum is called point of the local extremum.

Necessary condition of the existence of local extrema

If the point x_0 is the point of a local extremum of function f, and function is differentiable at this point, so there exists derivative f' at this point, then $f'(x_0) = 0$.

Consequence

Continuous function $f(x)$ can possess local extrema at points at which its first derivative equals zero - at so called **stationary points**, or at points at which the derivative does not exist. Commonly we speak about **critical points** of f (for the first derivative).

Sufficient condition of the existence of local extreme

Let function f be continuous at the point x_0 .

If there exists such left-hand neighbourhood of point x_0 , in which for $\forall x \in O^-(x_0)$ hold $f'(x) > 0$) ($f'(x) < 0$)
and such right-hand neighbourhood of point x_0 , in which for $\forall x \in O^+(x_0)$ hold $f'(x) < 0$) ($f'(x) > 0$),

then function f possesses strict local maximum (minimum) at the point x_0 .

Let $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Then point x_0 is a point of strict local extremum, which is

strict local maximum, if $f''(x_0) < 0$

strict local minimum, if $f''(x_0) > 0$.

Detection of local extrema

1. Critical points of the first derivative

2. The second derivative test

Global extrema

Let f be function defined on intervale J and let $x_0 \in J$. Function value $f(x_0)$ is called global maximum (minimum) of function f on interval J, if for all $x \in J$ hold $f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$).

Global maximum and global minimum are commonly called global extrema.

Detection of global extrema of continuous function defined on closed interval <*a*, *b*>

1. Find all critical points on open intervale (a, b)

   set $\{ x_1, x_2, \ldots, x_n \}$

2. Form set of function values at these points

   $FH = \{ f(x_1),\ f(x_2),\ \ldots, f(x_n), f(a),\ f(b) \}$

3. $max\ f(x) = max\ FH$, $min\ f(x) = min\ FH$