BASIC THEOREMS OF DIFFERENTIAL CALCULUS

Fermat theorem

If function f reaches at the point ξ its minimal or maximal value and it is differentiable at this point, i.e. there exists derivative of function at this point, then

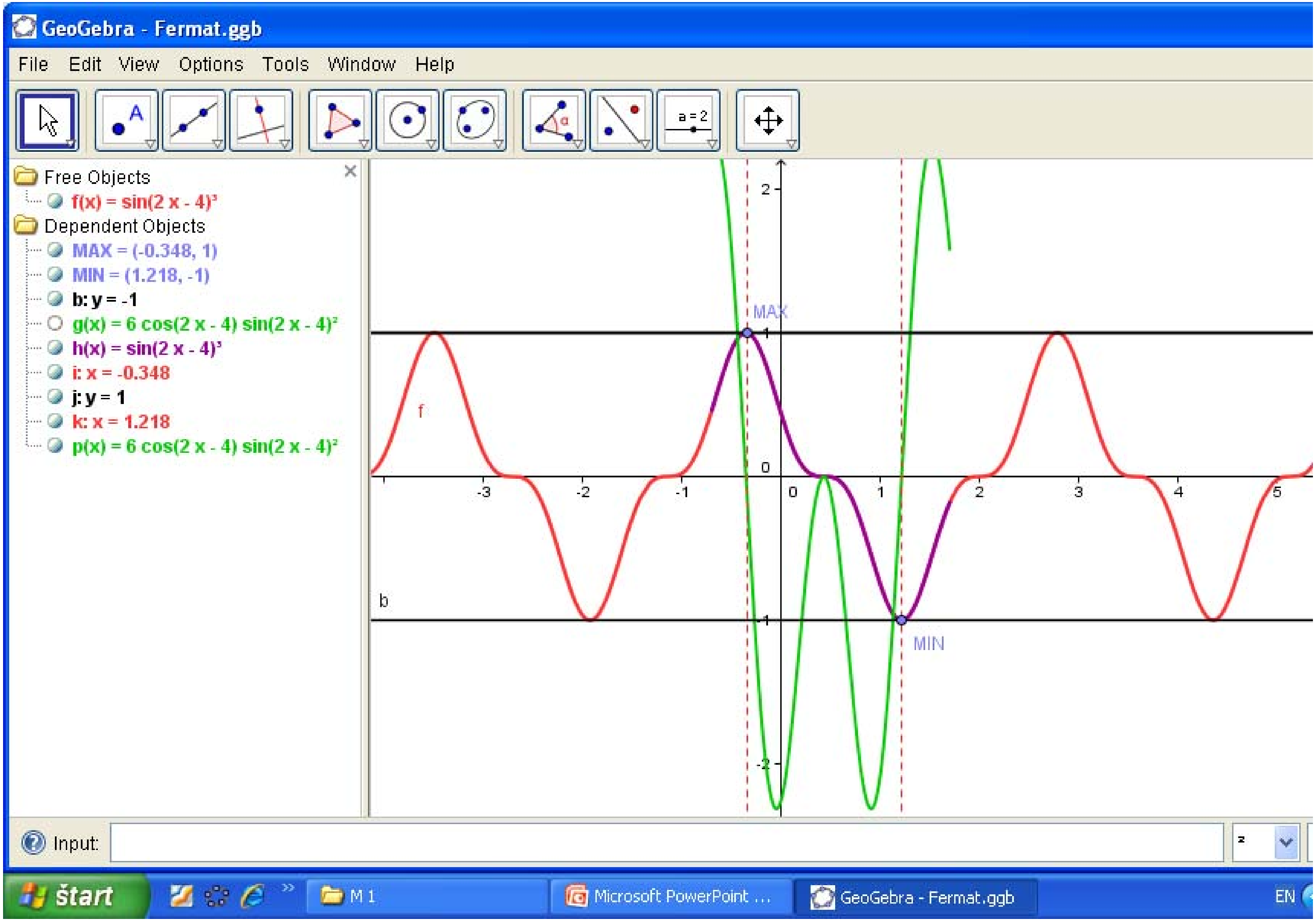
$$f'(\xi) = 0.$$

Geometric interpretation

Tangent line to the function graph at the point $[\xi, f(\xi)]$ is parallel to the axis x.

Slope of this tangent line is zero, and its equation is

$$y = f(\xi)$$



GeoGebra - Fermat.ggb
File Edit View Options Tools Window Help
Free Objects
f(x) = sin(2 x - 4)³
Dependent Objects
MAX = (-0.348, 1)
MIN = (1.218, -1)
b: y = -1
g(x) = 6 cos(2 x - 4) sin(2 x - 4)²
h(x) = sin(2 x - 4)³
i: x = -0.348
j: y = 1
k: x = 1.218
p(x) = 6 cos(2 x - 4) sin(2 x - 4)²
MAX
MIN
f
b
Input:
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M 1
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GeoGebra - Fermat.ggb
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Consequence of Fermat theorem

Function $f\,(x)$ can have extreme values at such points $x \in D(f)$, in which $f'(x) = 0$.

These points are called **stationary points** of function.

Rolle theorem

Let function f be

1. continuous on closed interval $\langle a, b \rangle$
2. differentiable at all points from open interval (a, b)

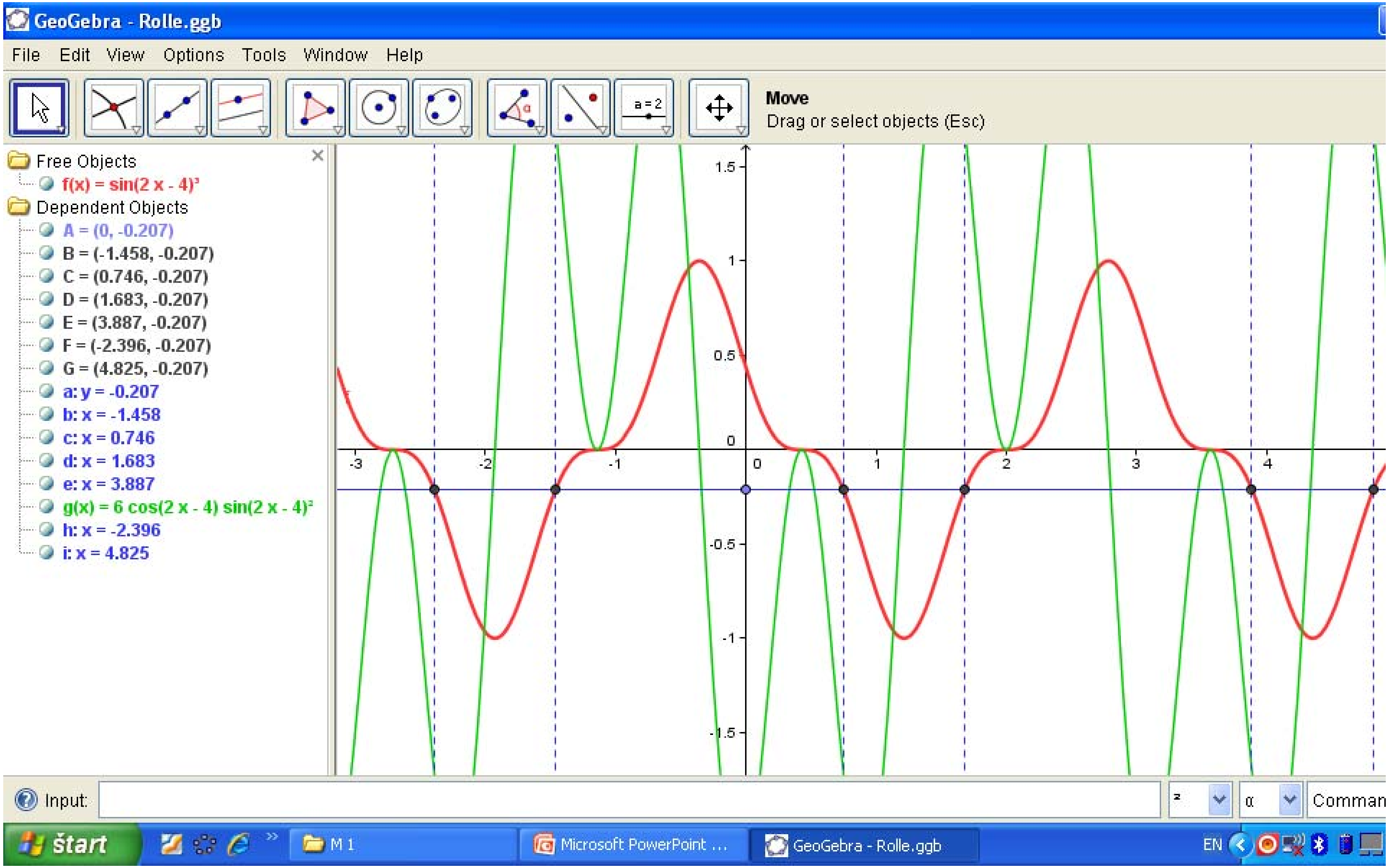
and let $f(a) = f(b)$. Then there exists at least one point $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Geometric interpretation

Continuous function $f(x)$ is monotone on interval (a, b).

As $f(a) = f(b)$, function is increasing on some parts and decreasing on other parts of this interval, therefore there exists at least one stationary point of function, which is the point of function extreme value.


GeoGebra - Rolle.ggb
File Edit View Options Tools Window Help
Move
Drag or select objects (Esc)
Free Objects
f(x) = sin(2 x - 4)³
Dependent Objects
A = (0, -0.207)
B = (-1.458, -0.207)
C = (0.746, -0.207)
D = (1.683, -0.207)
E = (3.887, -0.207)
F = (-2.396, -0.207)
G = (4.825, -0.207)
a: y = -0.207
b: x = -1.458
c: x = 0.746
d: x = 1.683
e: x = 3.887
g(x) = 6 cos(2 x - 4) sin(2 x - 4)²
h: x = -2.396
i: x = 4.825
Input:
štart
M 1
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GeoGebra - Rolle.ggb

Consequences

Slope of tangent line to the function graph is positive on some parts of the domain of definition and negative on other parts, while at the point $[\xi, f(\xi)]$, where tangent line is parallel to axis x , it equals zero, $f'(\xi) = 0$.

Sign of the function first derivative determines intervals in the domain of definition with different types of function monotonicity, its increasing or decreasing behaviour.

Lagrange theorem about function increment

Let *f* be function

1. continuous on closed interval <*a*, *b*>
2. differentiable at all points of open interval (*a*, *b*).

Then there exists at least one point $\xi \in (a, b)$ such, that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

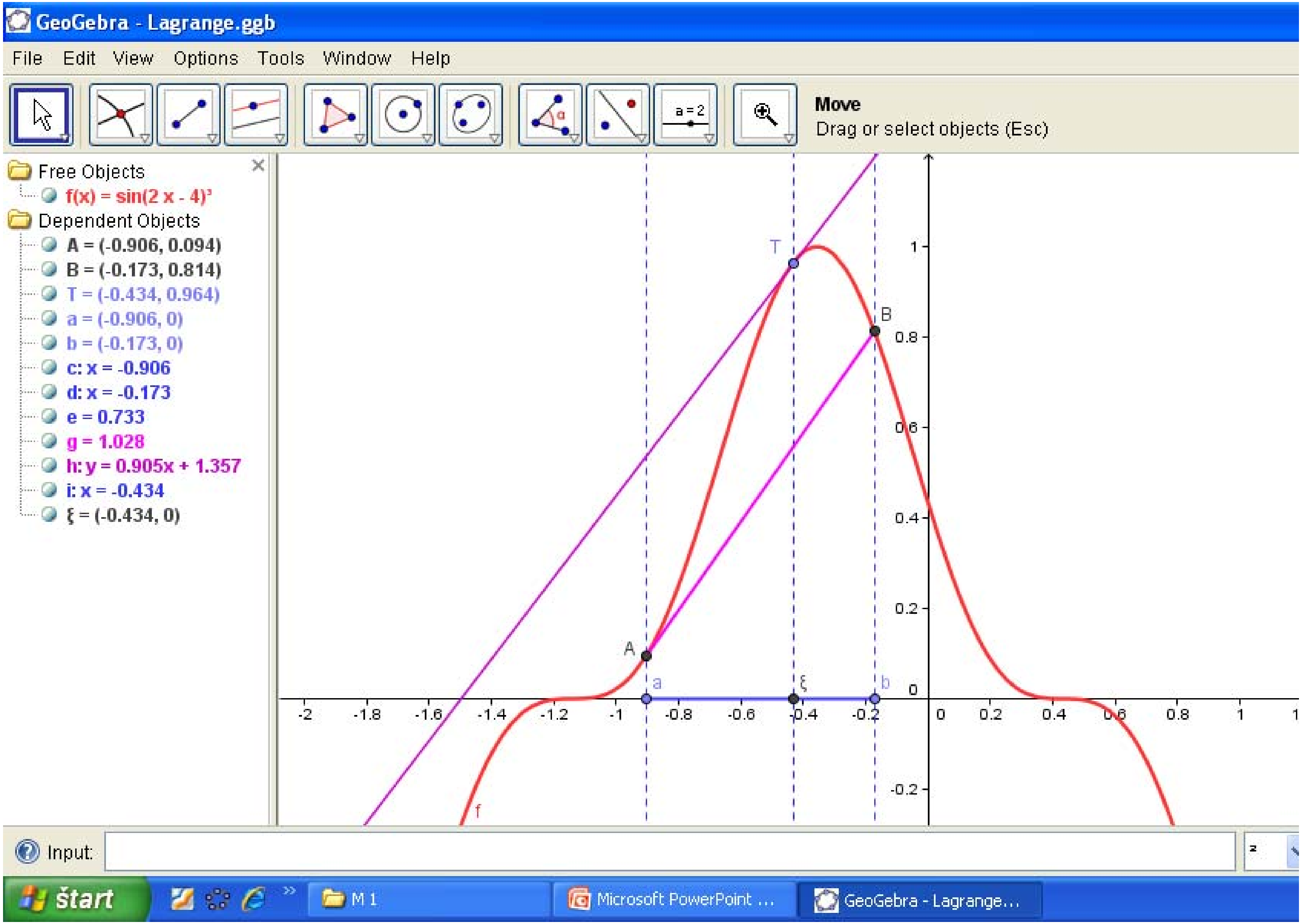
Geometric interpretation

Tangent line exists to the graph of function *f*(*x*) continuous on (*a*, *b*) at all points but the end points

A = [*a*, *f*(*a*)], *B* = [*b*, *f*(*b*)].

There exists at least one point $T = [\xi, f(\xi)]$ at the function graph such, that tangent line at this point is parallel to line segment *AB*, their slopes are equal

$$\frac{f(b) - f(a)}{b - a}$$


GeoGebra - Lagrange.ggb
File Edit View Options Tools Window Help
Move
Drag or select objects (Esc)
Free Objects
f(x) = sin(2 x - 4)³
Dependent Objects
A = (-0.906, 0.094)
B = (-0.173, 0.814)
T = (-0.434, 0.964)
a = (-0.906, 0)
b = (-0.173, 0)
c: x = -0.906
d: x = -0.173
e = 0.733
g = 1.028
h: y = 0.905x + 1.357
i: x = -0.434
ξ = (-0.434, 0)
Input:
štart
M 1
Microsoft PowerPoint ...
GeoGebra - Lagrange...

Physical interpretation

Function $s = f(t)$ represents trajectory of a point rectilinear motion, while its derivative $f'(t)$ determines velocity at time t.

Average (mean) velocity at time interval $<t_1, t_2>$ is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

At some moment $\xi \in <t_1, t_2>$ the instantaneous velocity equals to average (mean) velocity

$$f'(\xi) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Lagrange theorem is sometimes denoted as theorem about mean value.

If $f'(x) = 0$ for all $x \in (a, b)$, then function $f(x)$ is a constant function on this interval,

$f(x) = c, c \in R$, for $\forall\ x \in (a, b)$.

Consequence

If $f'(x) - g'(x) = 0$ for $\forall\ x \in (a, b)$, then

$f(x) = g(x) + c, c \in R, x \in (a, b)$.