Infinite number series

Let a sequence of real numbers be given as

$$\{a_n\} = a_1, a_2, ..., a_n, ...$$

Then, an expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$$

is called **infinite number series**, while a_n is the n-th term of the series.

Number

$$s_n = \sum_{i=1}^{n} a_n = a_1 + a_2 + ... + a_n , n = 1,2,...$$

is called the n-th partial sum of an infinite number series.

Number series is said to be **convergent**, if the sequnce $\{s_n\}$ of its partial sums has a proper limit

$$s = \lim_{n \to \infty} s_n$$

and number s is called the sum of the infinite number series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... = s$$

If the sequence $\{s_n\}$ does not have a proper limit,

then the related infinite number series is said to be **divergent**.

$$\sum_{n=1}^{\infty} n = 1+2+\ldots+n+\ldots, \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

$$\sum_{n=1}^{\infty} (-1)^n = -1+1-1+\ldots+(-1)^n+\ldots, \lim_{n\to\infty} s_n = does\ not\ exist$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots, \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots, \lim_{n\to\infty} s_n = \infty$$

Harmonic series

Necessary condition for the convergence of a number series

Let the infinite number series $\sum_{n=1}^{\infty} a_n$ be convergent, then $\lim_{n\to\infty} a_n = 0$.

If this condition is not valid for the terms of an infinite number series, then this series is necessarily divergent.

Majorant series to an infinite number series $\sum_{n=1}^{\infty} a_n$ is an infinite number series $\sum_{n=1}^{\infty} b_n$ such, that for any n the next inequality holds for their terms

$$b_n \geq | a_n | .$$

An infinite number series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if also the series $\sum_{n=1}^{\infty} |a_n|$ is convergent;

If the series $\sum_{n=1}^{\infty} |a_n|$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is relatively convergent.

Criteria for convergence

Rules for proving the convergence (divergence) of an infinite number series

1. Comparison test (criterion)

- Let $\sum_{n=1}^{\infty} a_n$ be such infinite number series, that there exist a convergent majorant number series $\sum_{n=1}^{\infty} b_n$; then also the series $\sum_{n=1}^{\infty} a_n$ is convergent.
- If the infinite number series $\sum_{n=1}^{\infty} a_n$ is divergent, then also its majorant number series $\sum_{n=1}^{\infty} b_n$ is divergent.

If the convergence of a given series is assumed, then it is enought to find a majorant convergent series (geometric series be the mostly used one).

If the divergence of a given series is assumed, then it is necessary to find a divergent series, for which the given series is majorant (harmonic series of one of its modifications is frequently used).

2. D´Alembert criterion for convergence

Let for an infinite number series $\sum_{n=1}^{\infty} a_n$ holds $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda$, then

- series is divergent for $\lambda > 1$,
- series is absolutely convergent for $\lambda < 1$,
- for $\lambda = 1$, we cannot decide about the convergence (divergence) of the number series.

3. Cauchy criterion for convergence

Let for an infinite number series $\sum_{n=1}^{\infty} a_n$ holds $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lambda$, then

- series is divergent $\lambda > 1$,
- series is absolutely convergent for $\lambda < 1$,
- for $\lambda = 1$, we cannot decide about the convergence (divergence) of the number series.

Alternating (oscilating) series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots, \forall n, a_n > 0 \ (a_n < 0)$$

Leibniz criterion for convergence

Let for all natural numbers n holds $0 < a_{n+1} \leq a_n$, then an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent if and only if $\lim_{n\to\infty} a_n = 0$.