Local, global and constrained extrema

1. Find all local extrema of function $f(x, y)$:

   a) $f(x,y) = x^3 - 3xy - y^3$ b) $f(x,y) = x^2 + y^2 - xy - x - y + 2$ c) $f(x,y) = x^3 + xy^2 - 27x$

   d) $f(x,y) = 2x^4 + y^4 - x^2 - 2y^2$ e) $f(x,y) = (y - x - 2)^2$ f) $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$

   g) $f(x,y) = y^3$ h) $f(x,y) = -\sqrt{x^2 + y^2}$ i) $f(x,y) = |x-1| + |y|$

Solution: a) $D(f) = E_2,\ f(-1,1) = 1,$ local maximum

b) $D(f) = E_2,\ f(1,1) = -1,$ local minimum

c) $D(f) = E_2,\ f(3,0) = -54,$ local minimum, $f(-3,0) = 54,$ local maximum

d) $D(f) = E_2,\ f(0,0) = 0,$ local maximum,
$f(0.5,1) = f(0.5,-1) = f(-0.5,1) = f(-0.5,-1) = -9/8,$ local minima

e) $D(f) = E_2,\ f(x, x+2) = 0,$ local minimum in all points on line $y = x + 2$

f) $D(f) = E_2,\ f(0,0) = 0,$ local minimum, $f(-1,0) = f(0,-1) = 2e^{-1},$ local maximum

g) $D(f) = E_2,$ no local extrema

h) $D(f) = E_2,\ f(0,0) = 0,$ local maximum

i) $D(f) = E_2,\ f(1,0) = 0,$ local minimum

2. Find constrained local extrema of functions on given constrains:

   a) $f(x,y) = x^3 - y^3,\ x + y - 3 = 0$ b) $f(x,y) = x + y,\ x^2 + y^2 - 1 = 0$

   c) $f(x,y) = xy - x + y - 1,\ x + y = 1$ d) $f(x,y) = -2x + 4y + 2,\ 4x^2 + y^2 = 17$

Solution: a) $f(1.5,1.5) = 22/7,$ constrained local minimum

b) $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2},$ constrained local minimum,
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2},$ constrained local maximum

c) $f(-0.5,1.5) = 0.25,$ constrained local maximum

d) $f(0.5,-4) = -15,$ constrained local minimum,
$f(\text{-0.5.4}) = 19,$ constrained local maximum

3. Find maximal and minimal value of function $f(x, y)$ on the given curve C:

   a) $f(x,y) = y^2 + 4x - x^2,\ C: y = \sqrt{9 - x^2},\ x \in \langle 0,3 \rangle$

   b) $f(x,y) = 2x + y,\ C: y = -\frac{1}{x},\ x \in \langle 1,2 \rangle$

   c) $f(x,y) = xy^2 + 4y,\ C: x = y^2,\ y \in \langle -2,0 \rangle$

d) $f(x,y)=x^2+y, C: y=\dfrac{2}{x}, x\in\langle 1,4\rangle$

e) $f(x,y)=x^2+4x-y^2, C: y=\sqrt{4-x^2}, x\in\langle 2,2\rangle$

f) $f(x,y)=x^2+y^2, C: y=x^2, x\in\langle -1,3\rangle$

Solution: a) $f_{\min}(x,y)=f(3,0)=3,\ f_{\max}(x,y)=f(1,\sqrt{8})=11$

b) $f_{\min}(x,y)=f(1,-1)=1,\ f_{\max}(x,y)=f(2,-0.5)=3.5$

c) $f_{\min}(x,y)=f(1,-1)=-3,\ f_{\max}(x,y)=f(4,-2)=8$

d) $f_{\min}(x,y)=f(1,2)=3,\ f_{\max}(x,y)=f(4,0.5)=16.5$

e) $f_{\min}(x,y)=f(-1,\sqrt{3})=-6,\ f_{\max}(x,y)=f(2,0)=12$

f) $f_{\min}(x,y)=f(0,0)=0,\ f_{\max}(x,y)=f(3,9)=90$

4. Find maximal and minimal value of function $f(x, y)$ on closed bounded set $M\subset E_2$:

a) $f(x,y)=x^2-2y^2+4xy-6x-1, M: x\ge 0, y\ge 0, y\le -x+3$

b) $f(x,y)=xy^2(4-x-y), M: x=0, y=0, x+y=6$

c) $f(x,y)=x^3+y^3-3xy, M:$ rectangular $ABCD, A=[0,-1], B=[2,-1], C=[2,2], D=[?,?]$

d) $f(x,y)=x^2-xy+y^2, M: |x|+|y|\le 1$

e) $f(x,y)=x^2+y^2-xy+x+y, M: x\le 0, y\le 0, x+y\ge -3$

f) $f(x,y)=3xy, M: x^2+y^2\le 2$

g) $f(x,y)=x^2+y^2-12x+16y, M: x^2+y^2\le 25$

h) $f(x,y)=x^2-y^2, M: x^2+y^2\le 4$

Solution: a) $f_{\min}(x,y)=f(0,3)=-19,\ f_{\max}(x,y)=f(0,0)=-1$

b) $f_{\min}(x,y)=f(2,4)=-64,\ f_{\max}(x,y)=f(1,2)=4$

c) $f_{\min}(x,y)=f(1,1)=f(A)=-1,\ f_{\max}(x,y)=f(B)=13$

d) $f_{\min}(x,y)=f(0,0)=0,\ f_{\max}(x,y)=f(1,0)=f(-1,0)=f(0,1)=f(0,-1)=1$

e) $f_{\min}(x,y)=f(-1,-1)=0,\ f_{\max}(x,y)=f(-3,0)=f(0,-3)=6$

f) $f_{\min}(x,y)=f(1,-1)=f(-1,1)=-3,\ f_{\max}(x,y)=f(1,1)=f(-1,-1)=3$

g) $f_{\min}(x,y)=f(3,-4)=-75,\ f_{\max}(x,y)=f(-3,4)=125$

h) $f_{\min}(x,y)=f(0,2)=f(0,-2)=-4,\ f_{\max}(x,y)=f(2,0)=f(-2,0)=4$