Partial derivatives, differential, tangent plane to graph of functions

1. Find all partial derivatives of function $f(x, y)$ up to the second order:

a) $f(x,y)=\frac{5}{x}+\frac{1}{(3y+2)^2}$ b) $f(x,y)=\arctan\frac{x-y}{1+xy}$ c) $f(x,y)=e^{\frac{x}{y}}+x^y$

d) $f(x,y)=\ln(\sin xy)$ e) $f(x,y)=\arccos\left[(x-y)^2\right]$ f) $f(x,y)=e^{\sin(x-y)}$

Solution: a) $f'_x=-5x^{-2},\ f'_y=-6(3y+2)^{-3},\ f''_{xx}=10x^{-3},\ f''_{yy}=54(3y+2)^{-4},\ f''_{xy}=f''_{yx}=0$

b) $f'_x=(1+x^2)^{-1},\ f'_y=-(1+y^2)^{-1},\ f''_{xx}=-2(1+x^2)^{-2},\ f''_{yy}=2(1+y)^{-2},\ f''_{xy}=f''_{yx}=0$

c) $f'_x=y^{-1}e^{\frac{x}{y}}+yx^{y-1},\ f'_y=-xy^{-2}e^{\frac{x}{y}}+x^y\ln x,$

$f''_{xx}=y^{-2}+(y^2-y)x^{y-2},\ f''_{yy}=xy^{-3}(2+xy^{-1})e^{\frac{x}{y}}+x^y\ln^2 x,$

$f''_{xy}=f''_{yx}=-y^{-3}e^{\frac{x}{y}}(2+xy^{-1})+x^{y-3}y(y^2-y+1)$

d) $f'_x=y\cot g(xy),\ f'_y=x\cot g(xy),\ f''_{xx}=-y^2\sin^{-2}(xy),\ f''_{yy}=-x^2\sin^{-2}(xy),$
$f''_{xy}=f''_{yx}=(\cos(xy)\sin(xy)-xy)\sin^{-4}(xy)$

e) $f'_x=2(y-x)(1+(x-y)^2)^{-\frac{1}{2}},\ f'_y=2(x-y)(1+(x-y)^2)^{-\frac{1}{2}},$
$f''_{xx}=f''_{yy}=-2(1+(x-y)^2)^{-\frac{3}{2}},\ f''_{xy}=f''_{yx}=2(1-(x-y)^2)(1+(x-y)^2)^{-\frac{3}{2}}$

f) $f'_x=\cos(x-y)e^{\sin(x-y)},\ f'_y=-\cos(x-y)e^{\sin(x-y)},$
$f''_{xx}=(\cos^2(x-y)+\sin(x-y))e^{\sin(x-y)},\ f''_{yy}=(\cos^2(x-y)-\sin(x-y))e^{\sin(x-y)},$
$f''_{xy}=f''_{yx}=(\sin(x-y)-\cos^2(x-y))e^{\sin(x-y)}$

2. Find all first partial derivatives of functions with three variables:

a) $f(x,y,z)=2^{yz}\arctan xz$ b) $f(x,y,z)=e^{xyz}\sin(x+y+z)$ c) $f(x,y,z)=\ln\frac{x+y}{z}$

Solution: a) $f'_x=2^{yz}z(1+x^2z^2)^{-1},\ f'_y=2^{yz}z\ln 2\arctan xz,\ f'_z=2^{yz}(x(1+x^2z^2)^{-1}+y\ln 2\arctan xz$

b) $f'_x=e^{xyz}(yz\sin(x+y+z)+\cos(x+y+z)),$
$f'_y=e^{xyz}(xz\sin(x+y+z)+\cos(x+y+z)),$
$f'_z=e^{xyz}(xy\sin(x+y+z)+\cos(x+y+z))$

c) $f'_x=f'_y=(x+y)^{-1},\ f'_z=-z^1$

3. Evaluate total differential of function $f(x, y)$ in the point A for given X:

a) $f(x,y)=x^2-2xy-3y^2,\ A=[-1,1],\ X=[1,2]$

b) $f(x,y)=e^{xy},\ A=[1,1],\ X=[2,2]$

c) $f(x,y)=\arctan xy,\ A=[1,2],\ X=[2,1]$

d) $f(x,y,z)=\frac{x^2-y^2}{z},\ A=[1,1,1],\ X=[0,0,1]$

e) $f(x,y,z)=xyz,\ A=[1,2,3],\ X=[2,3,4]$

Solution: a) $df_A(x,y) = -4(x+1) - 4(y-1), df_A(1,2) = -12$,

b) $df_A(x,y) = e(x-1) + e(y-1), df_A(2,2) = 2e$,

c) $df_A(x,y) = 2(x-1)/5 + (y-2)/5, df_A(2,1) = 1/5$

d) $df_A(x,y,z) = 2(x-1) - 2(y-1), df_A(0,0,1) = 0$

e) $df_A(x,y,z) = 6(x-1) + 3(y-2) + 2(z-3), df_A(0,0,0) = 11$

4. Find equation of tangent plane to the graph of function $f(x, y)$ in the point T:

a) $f(x,y) = 2x^2 + y^2, T = [1,1,?]$

b) $f(x,y) = x^4 + 2x^2y - xy + x, T = [1,?,2]$

c) $f(x,y) = \sqrt{9 - x^2 - y^2}, T = [2,-1,?]$

d) $f(x,y) = \sqrt{x^2 - y^2 + 1}, T = [1,1,?]$

e) $f(x,y) = xy, T = [?,2,2]$

f) $f(x,y) = e^{x^2+y^2-1}, T = [0,0,?]$

g) $f(x,y) = y\cos(3x+2y), T = [0,\pi,?]$

h) $f(x,y) = \ln(x^2 + y^2), T = [1,0,?]$

Solution: a) $4x + 2y - z - 3 = 0$

b) $5x + y - z - 3 = 0$

c) $2x - y + 2z - 9 = 0$

d) $x - y - z + 1 = 0$

e) $2x + y - z - 2 = 0$

f) $2x + 2y - e^{-1} z - 3 = 0$

g) $y - z = 0$

h) $2x - z - 2 = 0$