Linear differential equations of order 2 with constant coefficients

1. Find general solution of differential equation:
   a) $y'' - y' - 2y = 0$
   b) $y'' + 5y' + 6y = 0$
   c) $y'' - 4y' = 0$
   d) $y'' - 4y = 0$
   e) $y'' - 2y' + y = 0$
   f) $y'' + 6y' + 9y = 0$
   g) $y'' + 2y' + 5y = 0$
   h) $y'' + 4y = 0$

Solutions:
   a) $y = c_1 e^{2x} + c_2 e^{-x}, x \in R, c_1, c_2 \in R$
   b) $y = c_1 e^{-2x} + c_2 e^{-3x}, x \in R, c_1, c_2 \in R$
   c) $y = c_1 + c_2 e^{4x}, x \in R, c_1, c_2 \in R$
   d) $y = c_1 e^{2x} + c_2 e^{-2x}, x \in R, c_1, c_2 \in R$
   e) $y = c_1 e^{x} + c_2 x e^{x}, x \in R, c_1, c_2 \in R$
   f) $y = c_1 e^{-3x} + c_2 x e^{-3x}, x \in R, c_1, c_2 \in R$
   g) $y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x, x \in R, c_1, c_2 \in R$
   h) $y = c_1 \cos 2x + c_2 \sin 2x, x \in R, c_1, c_2 \in R$

2. Find general solution of differential equation using method of variation of constants:
   a) $y'' + 2y' + y = \dfrac{e^x}{x^2 + 1}$ b) $y'' + y = \dfrac{1}{\sin x}$ c) $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

Solutions:
   a) $y = e^{-x}(c_1 + c_2 x - \frac{1}{2}\ln(x^2 + 1) + x \operatorname{arctan} x), x \in R, c_1, c_2 \in R$
   b) $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sin x|, x \in (k\pi, (k+1)\pi), k \in Z, c_1, c_2 \in R$
   c) $y = e^{2x}(c_1 + c_2 x - x + x \ln|x|), x \in (-\infty, 0), \text{or } (0, \infty), c_1, c_2 \in R$

3. Find general solution of differential equation using the special form of the right side term:
   a) $y'' - y' - 6y = 4$ b) $y'' - 2y' + y = 3$ c) $y'' - 2y' = 5$

Solutions:
   a) $y = c_1 e^{3x} + c_2 e^{-2x} - \frac{2}{3}, x \in R, c_1, c_2 \in R$
   b) $y = c_1 e^{x} + c_2 x e^{x} + 3, x \in R, c_1, c_2 \in R$
   c) $y = c_1 + c_2 e^{2x} - \frac{5}{2}x, x \in R, c_1, c_2 \in R$

4. Find general solution of differential equation using the special form of the right side term:

a) $y'' + 4y = 2x + 4$ b) $y'' - 4y' + 4y = 4x^2 + 4x$ c) $y'' + y' = 6x - 1$

Solutions:

a) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x + 1, x \in R, c_1, c_2 \in R$

b) $y = c_1 e^{2x} + c_2 x e^{2x} + x^2 + 3x + \frac{5}{2}, x \in R, c_1, c_2 \in R$

c) $y = c_1 + c_2 e^{-x} + 3x^2 - 7x, x \in R, c_1, c_2 \in R$

5. Find general solution of differential equation using the special form of the right side term:

a) $y'' - y' - 2y = 4e^{-2x}$ b) $y'' + y = 5e^{3x}$

c) $y'' - 2y' = 4e^{2x}$ d) $y'' - 2y' + y = 4e^{x}$

Solutions:

a) $y = c_1 e^{-x} + c_2 e^{2x} + \frac{e^{-2x}}{4}, x \in R, c_1, c_2 \in R$

b) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}e^{3x}, x \in R, c_1, c_2 \in R$

c) $y = c_1 + c_2 e^{2x} + 2xe^{2x}, x \in R, c_1, c_2 \in R$

d) $y = c_1 e^{x} + c_2 x e^{x} + 2x^2 e^{x}, x \in R, c_1, c_2 \in R$

6. Find general solution of differential equation using the special form of the right side term:

a) $y'' - 4y' + 3y = (x+1)e^{-x}$ b) $y'' - 4y' + 3y = (x+1)e^{x}$

c) $y'' - 2y' + y = (x+1)e^{x}$

Solutions:

a) $y = c_1 e^{x} + c_2 e^{3x} + (\frac{1}{8}x + \frac{7}{32})e^{-x}, x \in R, c_1, c_2 \in R$

b) $y = c_1 e^{x} + c_2 e^{3x} (-\frac{1}{4}x^2 - \frac{3}{4}x)e^{x}, x \in R, c_1, c_2 \in R$

c) $y = c_1 e^{x} + c_2 x e^{x} + (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{x}, x \in R, c_1, c_2 \in R$

7. Find general solution of differential equation using the special form of the right side term:

a) $y'' + 4y' = \cos x + \sin x$ b) $y'' - 4y' + 5y = \sin 2x$

c) $y'' + 4y = \cos 2x$ d) $y'' - y' - 2y = e^{x} \sin x$

Solutions:

a) $y = c_1 + c_2 e^{-4x} - \frac{5}{17}\cos x + \frac{3}{17}\sin x, x \in R, c_1, c_2 \in R$

b) $y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x + \frac{8}{65}\cos x + \frac{1}{65}\sin x, x \in R, c_1, c_2 \in R$

c) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x \sin 2x, x \in R, c_1, c_2 \in R$

d) $y = c_1 e^{-x} + c_2 e^{2x} + e^{x}(-\frac{1}{10}\cos x - \frac{3}{10}\sin x), x \in R, c_1, c_2 \in R$

8. Find general solution and particular solution satisfying initial conditions:

a) $y'' + 2y' + 2y = \frac{1}{e^x \cos x}$, $y(0) = -1, y'(0) = 1$

b) $y'' - 2y' + y = e^{3x}$, $y(0) = \frac{5}{4}, y'(0) = \frac{7}{4}$

c) $y'' + y' = 4x + 1$, $y(0) = -1, y'(0) = -3$

d) $y'' + 2y' + 2y = \cos x$, $y(0) = \frac{1}{5}, y'(0) = -\frac{3}{5}$

Solutions:

a) general solution $y = e^{-x}(c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x)$, $x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi), k \in Z, c_1, c_2 \in R$

particular solution $y_p = e^{-x}(-\cos x + \cos x \ln|\cos x| + x \sin x), x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

b) general solution $y = c_1 e^x + c_2 x e^x + \frac{e^{3x}}{4}, x \in R, c_1, c_2 \in R$

particular solution $y_p = e^x + \frac{e^{3x}}{4}, x \in R$

c) general solution $y = c_1 + c_2 e^{-x} + 2x^2 - 3x, x \in R, c_1, c_2 \in R$

particular solution $y_p = 2x^2 - 3x + 1, x \in R$

d) general solution $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x + \frac{1}{5} \cos x + \frac{2}{5} \sin x, x \in R, c_1, c_2 \in R$

particular solution $y = -e^{-x} \sin x + \frac{1}{5} \cos x + \frac{2}{5} \sin x, x \in R$