Problems on ordinary differential equations

ODE with separated variables

1. Find general solution and particular solution satisfying initial condition

   a) $x + yy' = 0,\ y(0) = 1$

   b) $\frac{1}{x+1} + \frac{1}{y}y' = 0,\ y(1) = -1$

   c) $\frac{1}{x} - \frac{1}{y-1}y' = 0,\ y(-1) = 2$

   d) $1 + e^{y} y' = 0,\ y(1) = 0$

   e) $1 + \frac{1}{y+1}y' = 0,\ y(0) = 0$

   f) $x - 1 + \frac{1}{y}y' = 0,\ y(0) = 2$

   g) $1 + \frac{1}{y^2+1}y' = 0,\ y(0) = 1$

   h) $\frac{1}{x} + \frac{2y}{1-y^2}y' = 0,\ y(1) = -2$

   i) $\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{y}}y' = 0,\ y(0) = 1$

   j) $\tan x + \frac{1}{y}y' = 0,\ y(0) = 1$

Solutions:

a) $y_1 = \sqrt{c - x^2}, y_2 = -\sqrt{c - x^2}, x \in (-\sqrt{c}, \sqrt{c}), c > 0, y_P = \sqrt{1 - x^2}, x \in (-1,1)$

b) $y = \frac{c}{x+1}, x \in (-\infty,1)\ or\ x \in (-1,\infty), c \neq 0, y_P = \frac{-2}{x+1}, x \in (-1,\infty)$

c) $y = 1 + cx, x \in (-\infty,0)\ or\ x \in (0,\infty), c \neq 0, y_P = 1 - x, x \in (-\infty,0)$

d) $y = \ln(c - x), x \in (-\infty, c), c \in R, y_P = \ln(2 - x), x \in (-\infty,2)$

e) $y = -1 + ce^{-x}, x \in R, c \neq 0, y_P = -1 + e^{-x}, x \in R$

f) $y = ce^{-\frac{x^2}{2}+x}, x \in R, c \neq 0, y_P = 2e^{-\frac{x^2}{2}+x}, x \in R$

g) $y = \tan(c - x), x \in (c + (2k+1)\pi/2, c + (2k+3)\pi/2)$, $c \in R, k \in Z,$
$y_P = \tan(\pi/4 - x), x \in (-\pi/4, 3\pi/4)$

h) $y_1 = \sqrt{1 + cx}, y_2 = -\sqrt{1 + cx}, x \in \left(-\frac{1}{c}, 0\right), c > 0,\ or\ x \in (0,\infty), c > 0,$
$or\ x \in (-\infty,0), c < 0, or\ x \in \left(0, -\frac{1}{c}\right), c < 0, y_P = -\sqrt{1 + 3x}, x \in (0,\infty)$

i) $y = (c - \sqrt{x+1})^2, x \in (-1,\infty), c > 0, y_P = (2 - \sqrt{x+1})^2, x \in (-1,\infty)$

j) $y = c\cos x, x \in (-\pi/2 + k\pi, \pi/2 + k\pi)$, $c \neq 0, k \in Z,$
$y_P = \cos x, x \in (-\pi/2, \pi/2)$

ODE with separable variables

2. Find general solution and particular solution satisfying initial condition
   a) $2y - y' = 0,\ y(0) = -1$
   b) $3y - (x-1)y' = 0,\ y(0) = -2$
   c) $2e^{y} + y' = 0,\ y(2) = 0$
   d) $(1+y)x + x^2 y' = 0,\ y(-1) = 1$
   e) $\sqrt{y} - \sqrt{x}y' = 0,\ y(4) = 1$
   f) $y\cos x - \sin x y' = 0,\ y(\pi/2) = 2$
   g) $y\ln x + xy' = 0,\ y(1) = 1$
   h) $\dfrac{y^2+1}{x} - 2yy' = 0,\ y(1) = 1$
   i) $y^2 + 1 + xyy' = 0,\ y(1) = \sqrt{3}$
   j) $4 - y^2 + 2xyy' = 0,\ y(1) = -1$

Solutions:

a) $y = ce^{2x}, x \in R, c \in R, y_P = -e^{2x}, x \in R$

b) $y = c(x-1)^3, x \in R, c \in R, y_P = 2(x-1)^3, x \in R$

c) $y = -\ln(2x+c), x \in (-c/2, \infty), c \in R, y_P = -\ln(2x-3), x \in (3/2, \infty)$

d) $y = -1 + c/x, x \in (-\infty, 0), or\ x \in (0, \infty), c \neq 0, y_S = -1, x \in R,$
$y_P = -1 - 2/x, x \in (-\infty, 0)$

e) $y = (c + \sqrt{x})^2, x \in (0, \infty), c \in R,\ y_S = 0, x \in < 0, \infty), y_P = (\sqrt{x} - 1)^2, x \in (0, \infty)$

f) $y = c\sin x, x \in R, c \in R,\ y_P = 2\sin x, x \in R$

g) $y = e^{\frac{c}{x}}, x \in (-\infty, 0), or\ x \in (0, \infty), c \neq 0, y_S = 1, x \in R, y_P = 1, x \in R$

h) $y_1 = \sqrt{cx-1}, y_2 = -\sqrt{cx-1}, x \in (-\infty, 1/c), c < 0, or\ x \in (1/c, \infty), c > 0,$
$y_P = \sqrt{2x-1}, x \in (1/2, \infty)$

i) $y_1 = \sqrt{\dfrac{c}{x^2} - 1}, y_2 = -\sqrt{\dfrac{c}{x^2} - 1}, x \in (-\sqrt{c}, 0)\ or\ x \in (0, \sqrt{c}), c > 0,$
$y_P = \sqrt{\dfrac{4}{x^2} - 1}, x \in (0, 2)$

j) $y_1 = \sqrt{4 + cx}, y_2 = -\sqrt{4 + cx}, x \in (-\infty, -4/c), c < 0, or\ x \in (-4/c, \infty), c > 0,$
$y_{1S} = 2, x \in R, y_{2S} = -2, x \in R, y_P = -\sqrt{4 - 3x}, x \in (-\infty, 4/3)$

Linear differential equations of order 1

3. Find general solution and particular solution satisfying initial condition

a) $y' - y = e^{2x}$, $y(0) = 1$

b) $y' - 2xy = xe^{x^2}$, $y(0) = -1$

c) $y' + \dfrac{1}{x-1}y = e^x$, $y(0) = 1$

d) $y' + \dfrac{2}{x+1}y = (x+1)^2$, $y(0) = 1/5$

e) $y' - \dfrac{2y}{x} = \dfrac{3}{x^2}$, $y(1) = 2$

f) $y' + \dfrac{y}{\sqrt{x}} = \dfrac{1}{\sqrt{x}}$, $y(1) = 2$

g) $y' - y\cos x = \cos x$, $y(0) = -1$

h) $y' - \cot gx = \sin^3 x$, $y(\pi/2) = 0$

i) $y' - y\tan x = x\cos^2 x$, $y(0) = 1$

j) $y' + \dfrac{1}{x+1}y = \sin x$, $y(0) = 0$

Solutions:

a) $y = ce^x + e^{2x}, x \in R, c \in R, y_P = e^{2x}, x \in R$

b) $y = ce^{x^2} + \dfrac{x^2}{2}e^{x^2}, x \in R, c \in R, y_P = -e^{x^2} + \dfrac{x^2}{2}e^{x^2}, x \in R$

c) $y = \dfrac{c}{x-1} + \dfrac{x-2}{x-1}e^x, x \in (-\infty,1), or\ x \in (1,\infty), c \in R, y_P = \dfrac{x-1}{x-1}e^x, x \in (-\infty,1)$

d) $y = \dfrac{c}{(x+1)^2} + \dfrac{(x+1)^3}{5}, x \in (-\infty,-1), or\ x \in (-1,\infty), c \in R,$

$y_P = \dfrac{(x+1)^3}{5}, x \in (-1,\infty)$

e) $y = cx^2 - 1/x, x \in (-\infty,0)\ or\ x \in (0,\infty), c \in R, y_P = 3x^2 - 1/x, x \in (0,\infty)$

f) $y = ce^{-2\sqrt{x}} + 1, x \in (0,\infty), c \in R, y_P = e^{2-2\sqrt{x}}, x \in (0,\infty)$

g) $y = ce^{\sin x} - 1, x \in R, c \in R, y_P = -1, x \in R$

h) $y = c\sin x + \dfrac{x\sin x}{2} - \dfrac{\sin x \sin 2x}{4}, x \in (k\pi,(k+1)\pi), k \in Z, c \in R,$

$y_P = -\dfrac{\pi}{4}\sin x + \dfrac{x\sin x}{2} - \dfrac{\sin x \sin 2x}{4}, x \in (0,\pi)$

i) $y = c\cos x + x\sin x\cos x + \cos^2 x, x \in (-\pi/2 + k\pi, \pi/2 + k\pi), k \in Z, c \in R,$

$y_P = x\sin x\cos x + \cos^2 x, x \in (-\pi/2, \pi/2)$

j) $y = \dfrac{c - \cos x - x\cos x + \sin x}{x+1}, x \in (-\infty,-1)\ or\ x \in (-1,\infty), c \in R,$

$y_P = \dfrac{1 - \cos x - x\cos x + \sin x}{x+1}, x \in (-1,\infty)$