

Euclidean Space $\mathbf{E}_n$

Any ordered n -tuple of real numbers, $n \geq 1$

$$X = [x_1, x_2, \dots, x_n]$$

Can be regarded as a point in the n -dimensional space.

Numbers $x_1, x_2, \dots, x_n$ are called coordinates of point.

Let $X = [x_1, x_2, \dots, x_n]$, $Y = [y_1, y_2, \dots, y_n]$ are two arbitrary points in the n -dimensional space.

Distance of points X, Y is the following number

$$d(X, Y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2}$$

Set of all points in the n -dimensional space with defined distance of two arbitrary space points is called

n -dimensional Euclidean space - E_n

For any three points $X, Y, Z \in E_n$ holds:

1. $d(X, Y) \geq 0, d(X, Y) = 0 \Leftrightarrow X = Y$

2. $d(X, Y) = d(Y, X)$

3. $d(X, Y) \leq d(X, Z) + d(Z, Y)$

Function $d: E_n \times E_n \rightarrow \mathbf{R}$ is called **metric** of the space E_n ,

while pair (E_n, d) is called **metric space**.

Let $X_0 \in E_n$ be a point in the n -dimensional space and $\varepsilon > 0$ be a real number.

Set

$$O_\varepsilon(X_0) = \{ X \in E_n : d(X, X_0) < \varepsilon \}$$

is called **ε -neighbourhood of point X_0** .

Let us have a set $M \subset E_n$. Point $X_0 \in M$ is called **interior point** of the set M , if there exists $\varepsilon > 0$ such that ε -neighbourhood of point X_0 , is the subset of the set M

$$\exists \varepsilon > 0, O_\varepsilon(X_0) \subset M$$

Set of all interior points of the set M is called interior of the set M .

Set M is said to be **open**, if it consists of its interior points.

A point X_0 is called a **boundary point** of a set $M \subset E_n$, if any of its ε -neighbourhoods contains at least one point of the set M and at least one point which does not belong to the set M .

The set of all boundary points of the set M is called **boundary** of the set M .

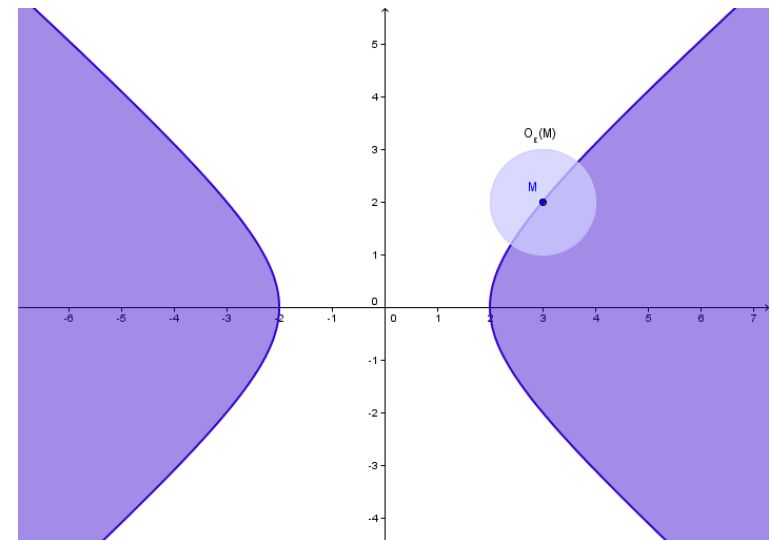
If we add to the set M all its boundary points, we receive the set which is called **closure** of the set M .

A set M is said to be **closed**, if it contains all its boundary points (i.e. it coincides with its closure).

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a \neq 0, b \neq 0$ is the boundary of

the set $M = \{X \in E_2; \frac{x^2}{a^2} - \frac{y^2}{b^2} \leq 1, a \neq 0, b \neq 0\}$,

All interior points of hyperbola are interior points of the set M , which is closed.

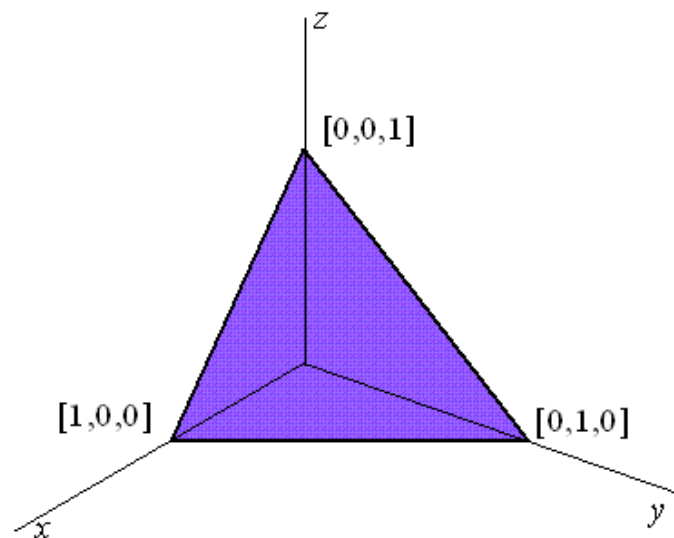
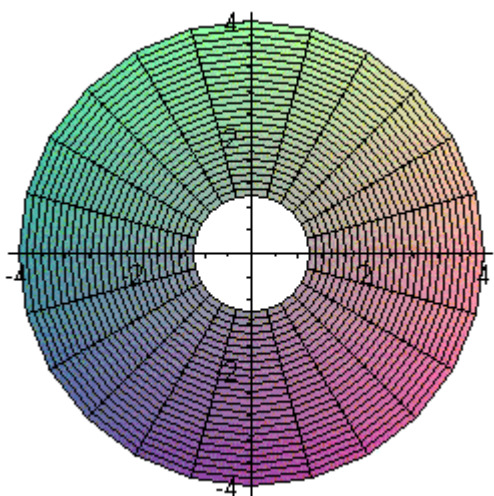


A set M is said to be **connected**, if each pair of points can be joined (connected) by a simple curve, which is all included in the set M .

Any opened connected set $M \subset E_n$ is called **region**.

Region $M \subset E_n$ is said to be **simply connected**, if any bounded region with the boundary in a simple closed curve $k \subset M$, is a part of it. Simply connected region does not contain "holes".

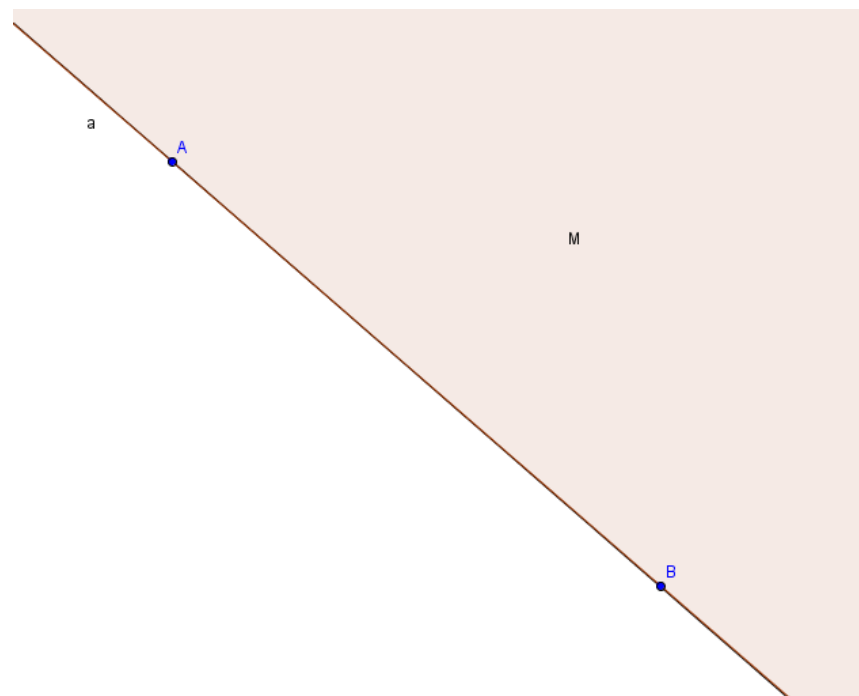
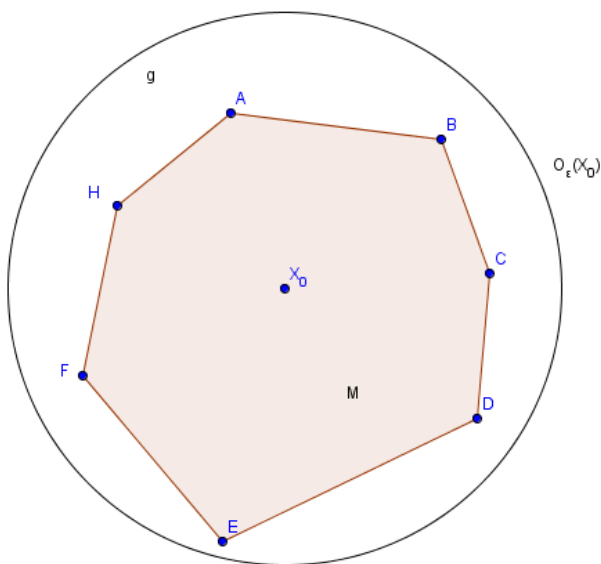
Annulus is not a simply connected region, triangle is a simply connected region



A set M is said to be **bounded**, if there exists a real number $k > 0$ and such point $X_0 \in M$ that for any point $X \in M$ holds

$$d(X, X_0) < k.$$

n -gon is a bounded set, semi-plane is not a bounded set



A point is called **isolated point** of the set $M \subset E_n$, if none of its ε -neighbourhoods contains no other point of the set M .

A point is called **limit point** of the set $M \subset E_n$, if any of its ε -neighbourhoods contains infinitely many points of the set M .

Any bounded infinite set $M \subset E_n$ has at least one limit point. A limit point is either an interior or a boundary point of the set.

An interior point of the set is always an element of the set, an exterior point never belongs to the set.

A boundary point of the set can belong to the set, but it is not necessary a point of the set.

Let a non-empty set $M \subset E_n$ be given and an arbitrary point $X_0 \in E_n$. Then exactly one of the following holds:

1. Point X_0 belongs to the set M with at least one of its neighbourhoods $O_\varepsilon(X_0) \subset M$. Point X_0 is an interior point of the set M .
2. Point X_0 does not belong to the set M neither does any point from its arbitrary neighbourhood, $O_\varepsilon(X_0) \cap M = \emptyset$. Point X_0 is an exterior point of the set M .
3. Any neighbourhood $O_\varepsilon(X_0)$ of the point X_0 contains at least one point of the set M and at least one point that does not belong to the set M . Point X_0 is a boundary point of the set M .
4. There exists such neighbourhood $O_\varepsilon(X_0)$ of the point X_0 , for which $O_\varepsilon(X_0) \cap M = \{X_0\}$. Point X_0 is isolated and boundary point of the set M .

5. Any neighbourhood $O_\varepsilon(X_0)$ of the point X_0 contains infinitely many point of M . Point X_0 is a limit point of the set M .

Any set of points in E_n which is opened and connected is called a **region**.

A set of points in E_n which is the closure of a region is called a **closed region**.

Any closed region can be obtained from a suitable region by adding all its boundary points.

A set M is closed if and only if it contains all its limit points.