Multiple integrals

Multiple integral of function $f(X)$ with more variables on arbitrary measurable closed region $G \subset E_n$, $n \geqq 1$.

Region G is measurable, if there exists its measure, number denoted as $\mu(G)$.

Let region G can be devided into n partial measurable not overlaping subregions G_i with measures denoted as $\mu(G_i)$, while

1. $G_i \cap G_j = \emptyset$**, for all** $i \neq j$, $i, j = 1, \ldots, n$
2. $\cup\, G_i = G$**, for** $i = 1, \ldots, n$

If for any sequence of integral sums of function *f*(*X*) on *G*

$$\sum_{i=1}^{n} f(\mathrm{X}_i).\mu(G_i)$$

a unique proper limit exists, then this number is called multiple

integral of function *f* on region *G* and is denoted

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathrm{X}_i).\mu(G_i) = \int_G f(X)dX$$

and function *f*(*X*) is said to be integrable on region $G \subset E_n$.

Function *f* integrable on a measurable closed region $G \subset E_n$ is bounded on this region.

Function *f* continuous (up to finite number of points) on a measurable closed region $G \subset E_n$ is integrable on this region.

Properties of multiple integrals

1. Linearity

If functions f_1, f_2, ..., f_k are integrable on a region G and c_1, c_2, ..., c_k are real numbers, then

$$\int_G (c_1 f_1(X) + c_2 f_2(X) + \ldots + c_k f_k(X))dX =$$

$$= c_1 \int_G f_1(X)dX + c_2 \int_G f_2(X)dX + \ldots + c_k \int_G f_k(X)dX$$

2. **Additivity**

Let function f be integrable on a region G and let

$$G = \bigcup_{i=1}^{k} G_i$$

where G_i are measurable regions with no common interior points.

Then

$$\int_G f(X)dX = \sum_{i=1}^{k} \int_{G_i} f(X)dX$$

3. Monotonicity

Let functions *f*, *g* are integrable on a region *G* and let for $\forall X \in G$ holds $f(X) \le g(X)$, then

$$\int_G f(X)dX \le \int_G g(X)dX$$

4. Positivity

Let function *f* be integrable on a measurable region *G* and let $f(X) \geqq 0$ for all $X \in G$, then

$$\int_G f(X)dX \ge 0$$

5. Let function f be integrable on a measurable region G ,

then also funkcia | $f(X)$ | is integrable on G and

$$\left|\int_G f(X)\,dX\right| \le \int_G |f(X)|\,dX$$