Constrained local extrema of functions with two variables

Let f be function with two variables defined on $D(f) \subset E_2$ and let set $V = \{[x, y] \in D(f): g(x, y) = 0\} \subset D(f)$ be given.

Condition determined by equation

$$g(x, y) = 0$$

which is satified by coordinates of all points from the function f domain of definition $D(f)$ that are in the set V is called constraint.

Extrema of function f, attained on the set $V \subset D(f)$ determined by constraint are constrained local extrema of function f.

Point $A = [x_0, y_0]$ is called the point of constrained local maximum (minimum) of function f for the constraint $g(x, y) = 0$, if there exists such neighbourhood $O_\varepsilon(A)$ of point A, that for all $X \in O_\varepsilon(A)$, whose coordinates satisfy given constraint holds

$$f(X) \leqq f(A) \ (\ (f(X) \geqq f(A) \ ).$$

In case of strict inequalities we speak about strict constrained local maximum or minimum.

Constrained local minimum and maximum of function are together called constrained local extrema of function.

How to determine constrained local extrema of function $f(x, y)$

1. Variable y can be extracted from the constraint $g(x, y) = 0$ and determined as function of variable x

$$y = h(x)$$

This function can be substituted to the function $f(x, y)$,

while a composite function of one variable x defined on the set V can be obtained

$$f(x, h(x)) = F(x)$$

All local extrema of function $F(x)$ on set V are also constrained local extrema of function $f(x, y)$ with two variables on set V.

2. Variable x can be extracted form the constraint $g(x, y) = 0$ and determined as function of variable y

$$x = h(y)$$

This function can be substituted to the function $f(x, y)$,

while a composite function of one variable y defined on the set V can be obtained

$$f(h(y), y) = F(y)$$

All local extrema of function $F(y)$ on set V are also constrained local extrema of function $f(x, y)$ with two variables on set V.

3. In case, none from variables x or y can be extracted from the constraint $g(x, y) = 0$ and expreessed in terms of the other, the method of Lagrange multipliers can be used.

We define an auxhiliary function called Lagrange function

$$F(x, y) = f(x, y) + \lambda\, g(x, y),$$

where λ is an arbitrary constant called Lagrange multiplier.

Function $F(x, y)$ is defined on set $D(f)$,

and moreover, in all points of the set V holds

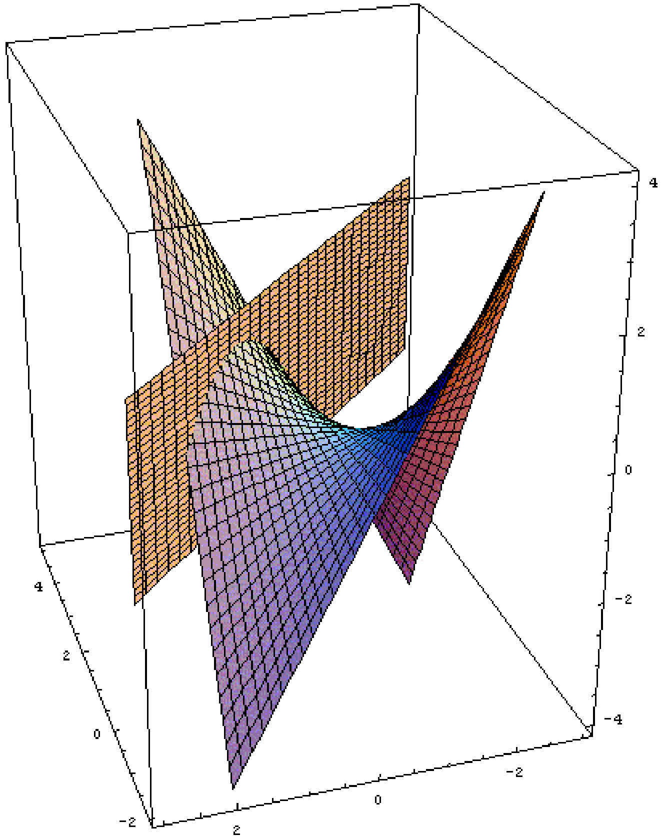
$$F(x, y) = f(x, y),$$

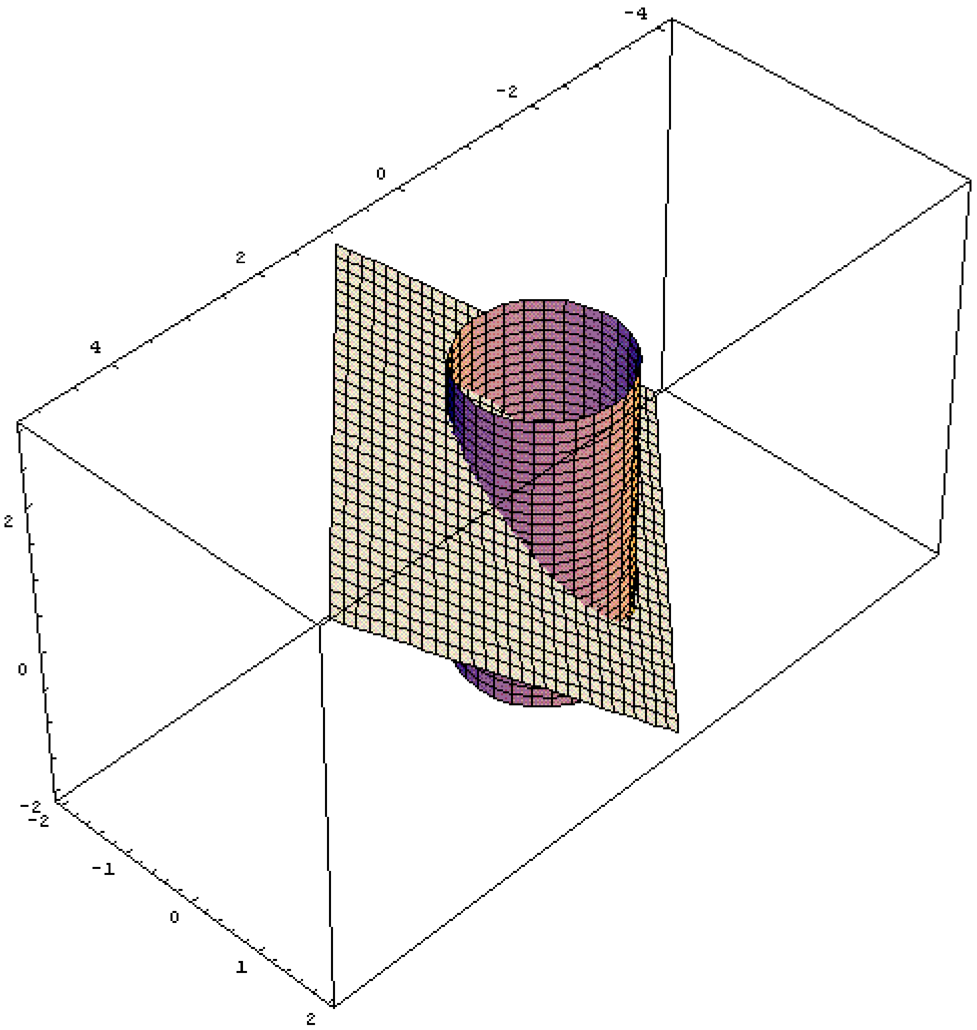
as $g(x, y) = 0$ in the points of set V.

If any point $A = [x_0, y_0] \in V$ is the point of local extremum of function $F = f + \lambda g$, then point A is the point of constrained local extremum of function f for the constraint $g(x, y) = 0$.

Geometric interpretation

Constrained local extrema of function *f* are *z*-coordinates of extremely located points on curve, which is intersection of graph *G*(f) of function *f* with the cylindrical surface determined by curve defined in the plane xy by constraint, while lines on this surface are in direction of coordinate axis *z*.

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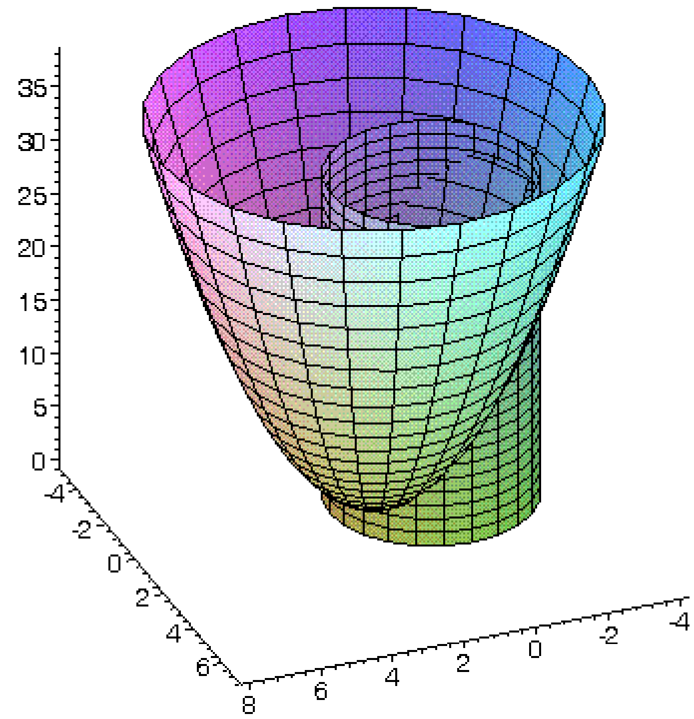
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Global extrema of function with more variables

Let f be function with n variables, $n \geq 1$,

defined on the set $M \subset D(f)$.

Maximum (minimum) of the set $H(f)$, which is the range of function f for all $X \in M$ is called the global (absolute) maximum (minimum) of function f on the set M.

Global maximum and minimum are called together global extrema of function f on the set M.

If M is an open region, function f may not attain any global extrema on this set.

If M is closed bounded set and f is a function continuous on M, then the global extrema on M are attained, and they can be found in the following steps:

1. We find all local extrema of function f inside set M, while it is sufficient to find values at all critical interior points of M.

2. We find all local extrema of function f on the boundary of set M, which are constrained local extrema of function f on boundary of M.

3. Global maximum (minimum) of function f on set M is then the greatest (least) from all found values, local extrema of f inside set M and constrained local extrema of f on boundary of set M.