Partial derivatives of higher orders of functions with two variables

Partial derivatives of partial derivatives of function with two variables *f*(*x*, *y*) are called second partial derivatives, or partial derivatives of the second order.

$$f''_{xx} = [f'_x]'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f''_{xy} = [f'_x]'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f''_{yx} = [f'_y]'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad f''_{yy} = [f'_y]'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

If partial derivatives of the second order of function *f* are continuous functions, then

$$f''_{xy} = f''_{yx}.$$

Function *f* of *n* variables can have up to n^2 partial derivatives of the second order.

For n = 2 there exist 2^2 = 4 partial derivatives of the second order. Partial derivatives of partial derivatives of the second order are called third partial derivatives or partial derivatives of the third order.

$$f'''_{xxx} = [\,f''_{xx}\,]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} \qquad f'''_{xxy} = [\,f''_{xx}\,]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

$$f'''_{xyx} = [\,f''_{xy}\,]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x} \qquad f'''_{xyy} = [\,f''_{xy}\,]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x \partial y \partial y}$$

$$f'''_{yxx} = [\,f''_{yx}\,]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y \partial x \partial x} \qquad f'''_{yxy} = [\,f''_{yx}\,]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y \partial x \partial y}$$

$$f'''_{yyx} = [\,f''_{yy}\,]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^2 \partial x} \qquad f'''_{yyy} = [\,f''_{y}\,]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}$$

Number of partial derivatives of function with two variables of the third order is $2^2 \times 2 = 2^3 = 8$.
Partial derivatives of partial derivatives of order *n*-1 are called *n*-th partial derivatives or partial derivatives of order *n*.

$$f^{(n)}_{\underbrace{xx\ldots x}_{n}} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-1}}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial x^n} \qquad f^{(n)}_{\underbrace{xx\ldots x}_{n-1} y} = [f^{n-1}_{\underbrace{xx\ldots x}_{n-1}}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{(n-1)} f}{\partial x^{n-1}}\right) = \frac{\partial^{(n)} f}{\partial x^{n-1}\partial y}$$

$$f^{n}_{\underbrace{xx\ldots x}_{n-2} yx} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-2} y}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y}\right) = \frac{\partial^n f}{\partial x^{n-2}\partial y\partial x} \qquad f^{n}_{\underbrace{xx\ldots x}_{n-2} yy} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-2} y}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y}\right) = \frac{\partial^n f}{\partial x^{n-2}\partial y^2}$$

…

$$f^{n}_{\underbrace{yy\ldots y}_{n-1} x} = [f^{(n-1)}_{\underbrace{yy\ldots y}_{n-1}}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial y^{n-1}}\right) = \frac{\partial^n f}{\partial y^{n-1}\partial x} \qquad f^{n}_{\underbrace{yy\ldots y}_{n}} = [f^{(n-1)}_{\underbrace{yy\ldots y}_{n-1}}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial y^{n-1}}\right) = \frac{\partial^n f}{\partial y^n}$$

Number of partial derivatives of function with two variables of order *n* is $2^{n-1} \times 2 = 2^n$.

By means of partial derivatives of higher orders we can investigate properties of functions with more variables, which means

- **determine critical points from the function domain of definition $D(f)$**
- **identify existence of function local extrema – min/max values**
- **investigate special points on function graph, e.g. saddle point**

Local extrema
of functions with two variables

Let f be a function with more variables defined on $D(f) \subset \mathrm{E}_n$ and let A be an arbitrary point from its domain of definition. Function f is said to have local minimum, (local maximum) $f(A)$ in the point A, if there exist such neighbourhood $O_\varepsilon(A)$ of point A that for all points $X \in D(f) \cap O_\varepsilon(A)$ holds

$$f(X) \geqq f(A) \qquad (f(X) \leqq f(A)).$$

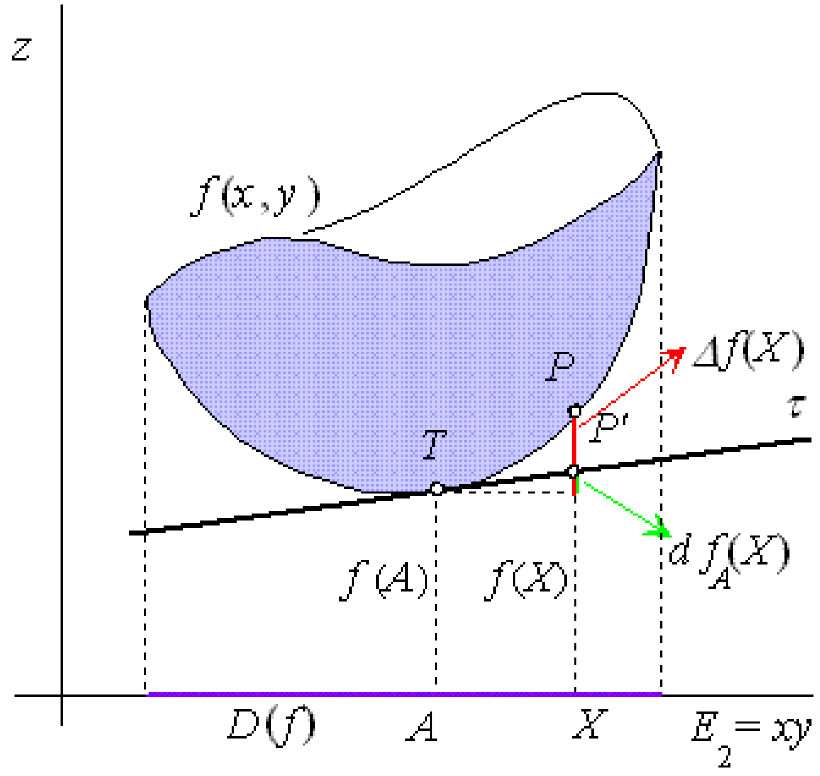
In case of sharp inequalities we speak about sharp local minimum or sharp local maximum.

Local minimum and local maximum of function are called together local extrema of function.

Function *f* can reach local extrema in the following points only:

1. **stationary points, in which all partial derivatives, if they exist, are equal to zero**
2. **points, in which no partial derivative exists.**

Total differential of function *f* equals zero in function stationary point.



Geometric interpretation for function with two variables

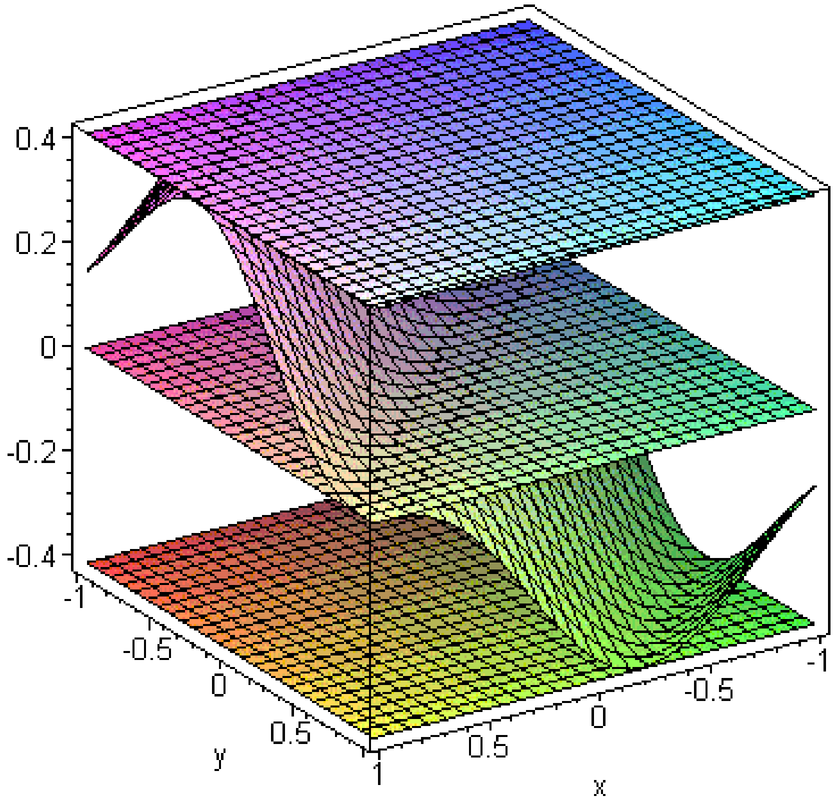
Let $[x_0, y_0] \in D(f)$ be stationary point of function $f(x, y)$,
Then graph of function f, surface $G(f) \subset E_3$, has tangent plane parallel to coordinate plane xy in point $T = [x_0, y_0, f(x_0, y_0)]$ with equation

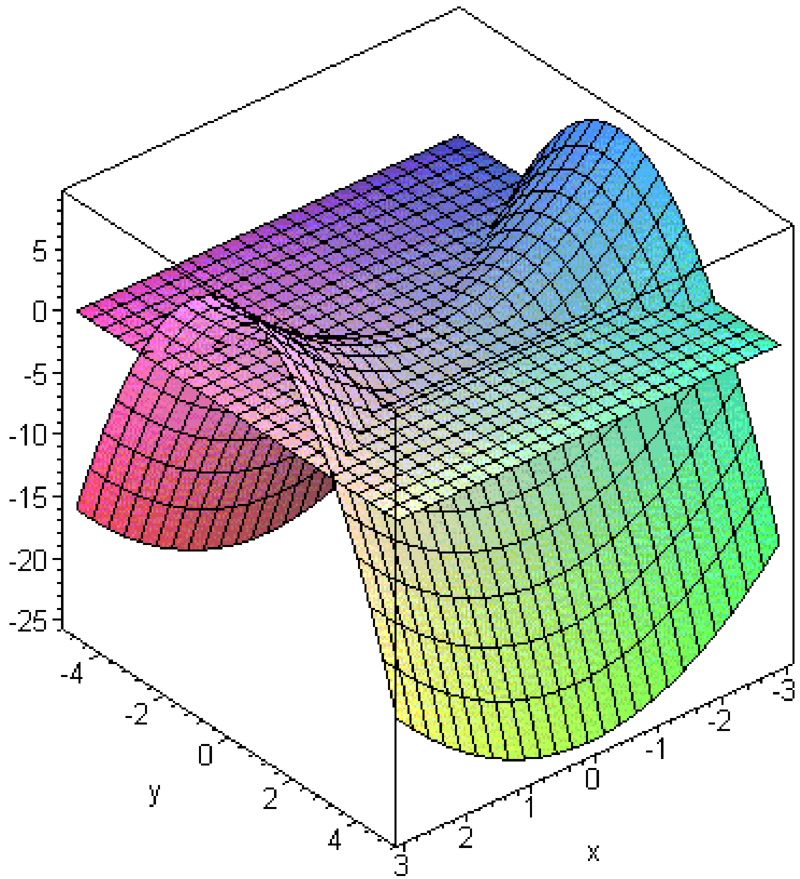
$$z = f(x_0, y_0)$$

Stationarity of point A is necessary but not satisfactory condition for the existence of local extreme of function in this point.

Function can reach local extrema also in such points, in which it is not differentiable.

Local maximum and minimum **Saddle point, which is not a point of local extreme**





Let $A = [x_0, y_0]$ be stationary point of function f with two variables, and let there exist continuous second partial derivatives of function f in some neighbourhood $O_\varepsilon(A)$ of point A. Then:

a) if

$$D(x_0, y_0) = \begin{vmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{yx}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{vmatrix} > 0$$

function has a sharp local extrem $f(x_0, y_0)$ in the point A, which is

- sharp local minimum, if $f''_{xx}(x_0, y_0) > 0$, or $f''_{yy}(x_0, y_0) > 0$**,**

- sharp local maximum, if $f''_{xx}(x_0, y_0) < 0$, or $f''_{yy}(x_0, y_0) < 0$**.**

b) if $D(x_0, y_0) < 0$, function f does not have sharp local extreme in the point A.

c) if $D(x_0, y_0) = 0$, we cannot decide about the existence of extreme of function f in the point A.

Determinant D is called

Hesse determinant of function with two variables $f(x, y)$ in point A.

Investigation of local extrema of function with two variables:

1. **Find all stationary points of function and points, in which function does not have partial derivatives.**
2. **Analyze all critical points from step 1 and possible existence of function extrema in these points.**
   - **In points, in which partial derivatives do not exist, existence of local extreme can be proved from definition.**
   - **In stationary points, in which second partial derivatives are continuous, Hesse determinant of function in this point can be used to decide about existence of local extrema.**
   - **In case of zero Hesse determinant in the stationary point, behaviour of function on the neighbourhood of this point must be investigated by means of definition of local extrema.**