Partial derivatives
of functions in two variables

Let $z = f(x, y)$ be a function defined on certain neighbourhood of point $A = [x_0, y_0]$, which is the limit point of its domain of definition $D(f)$.
Let us determine set

$$M_x = \{x \in R : [x, y_0] \in D(f)\} \subset \boldsymbol{E}_1$$

and define function

$$g : M_x \to R \quad \forall x \in M_x \quad g(x) = f(x, y_0)$$

Derivative $g'(x_0)$ of function $g(x)$ in the point x_0, if it exists, is said to be partial derivative of function $f(x, y)$ in the point $A = [x_0, y_0]$ with respect to variable x, denoted

$$f'_x(x_0, y_0) = f'_x(A) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(A)$$

$$f'_x(x_0, y_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Analogously we can determine set

$$M_y = \{y \in R : [x_0, y] \in D(f)\} \subset \boldsymbol{E}_1$$

and define function

$$h : M_y \to R \quad \forall y \in M_y \quad h(y) = f(x_0, y)$$

Derivative $h'(y_0)$ in the point y_0 of function $h(y)$, if it exists, is partial derivative of function $f(x, y)$ in the point $A = [x_0, y_0]$ with respect to variable y, denoted

$$f'_y(x_0, y_0) = f'_y(A) = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(A)$$

$$f'_y(x_0, y_0) = \lim_{y \to y_0} \frac{h(y) - h(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

Function $f(x, y)$ continuous in the point A may have no partial derivatives defined in the point A.

Function $f(x, y)$, whose partial derivatives in the point A exist, need not be continuous in the point A.

Let $M \subset E_2$ be set of such point from $D(f)$, in which there exist partial derivatives of function $f(x, y)$ with respect to x (y).
Function determined on set M, in which any point $A \in M$ is attached partial derivative of function $f(x, y)$ in the point A with respect to x (y), is said to be partial derivative of function $f(x, y)$ with respect to x (y)

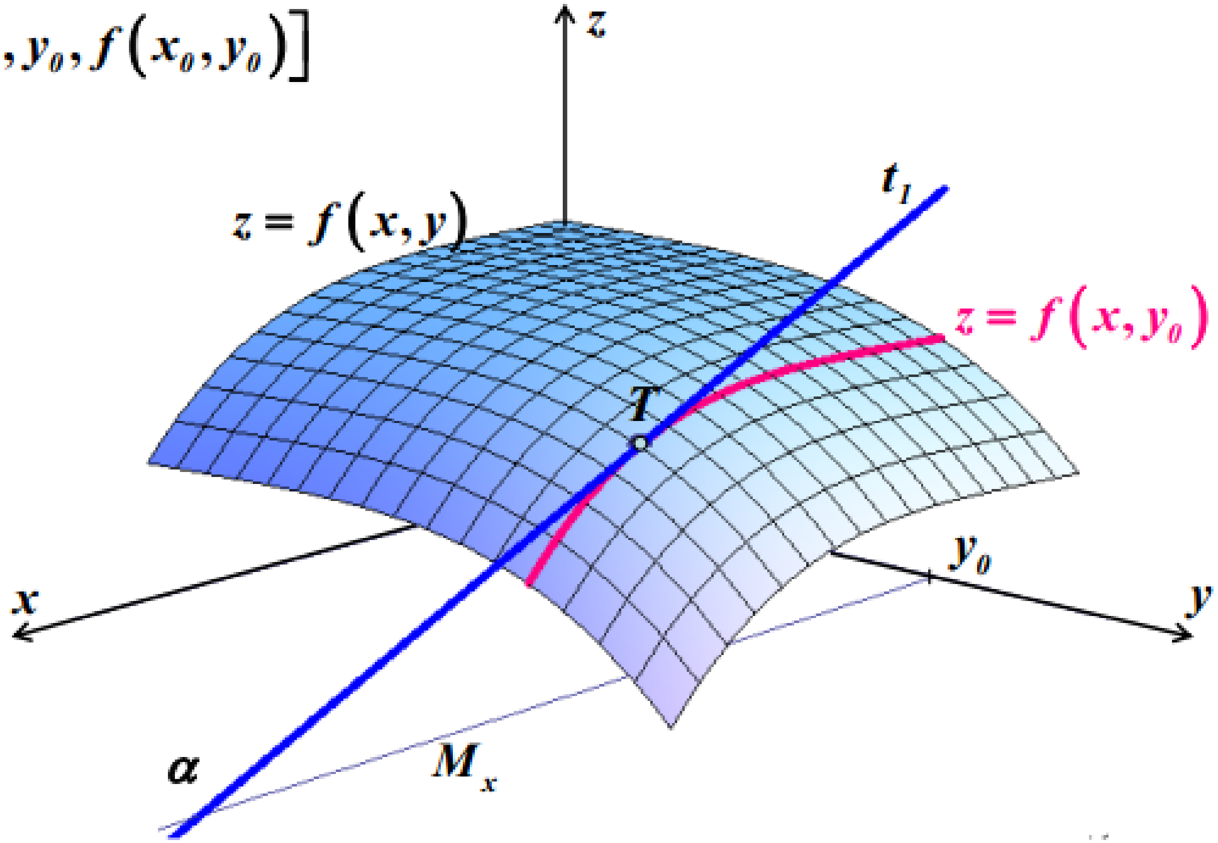
$$f'_x = \frac{\partial f}{\partial x} \qquad\qquad f'_y = \frac{\partial f}{\partial y}$$

$$f'_x : M \to R \qquad\qquad f'_y : M \to R$$

$$f'_x : A \to f'_x(A) \qquad\qquad f'_y : A \to f'_y(A)$$

Geometric meaning of partial derivatives

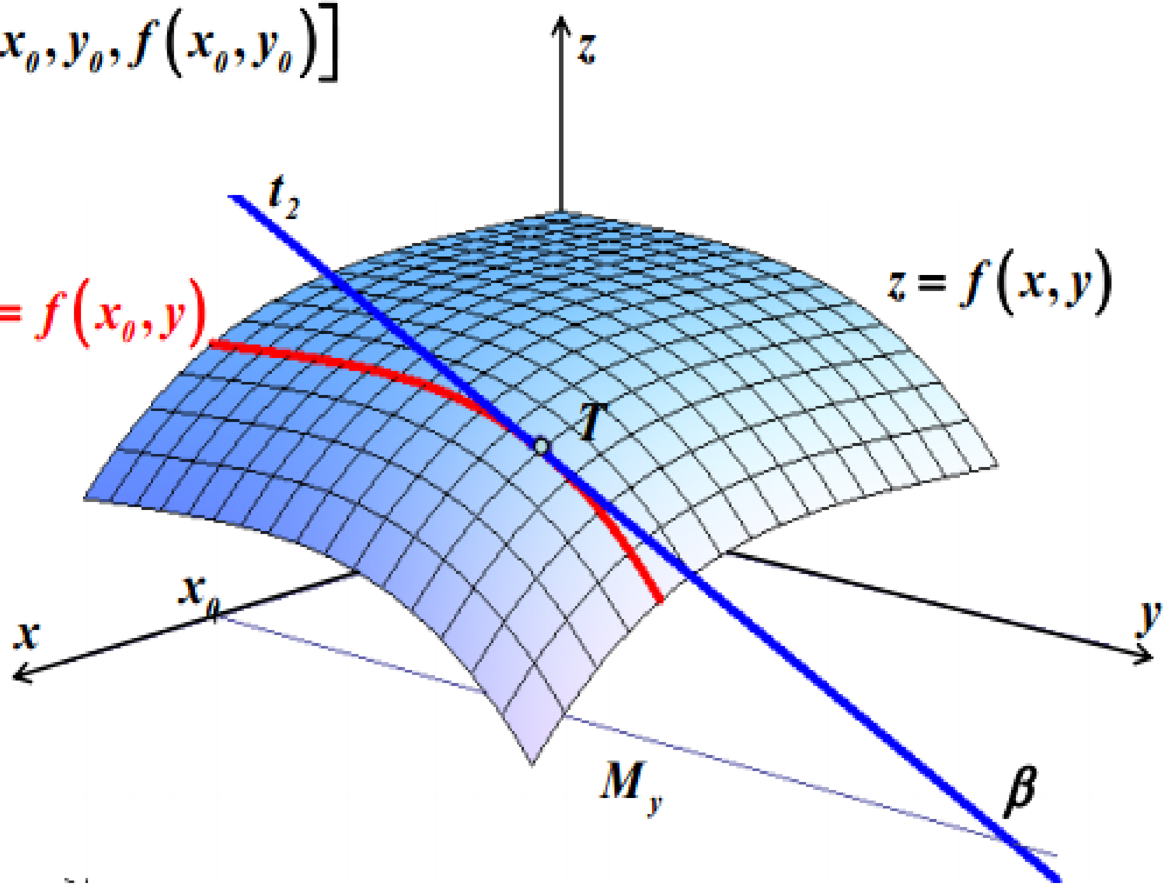
$$T = \left[x_0, y_0, f\left(x_0, y_0\right)\right]$$



$$tg\alpha = f'_x\left(x_0, y_0\right)$$

$$\vec{s}_{t_1} = \left(1, 0, f'_x\left(x_0, y_0\right)\right)$$

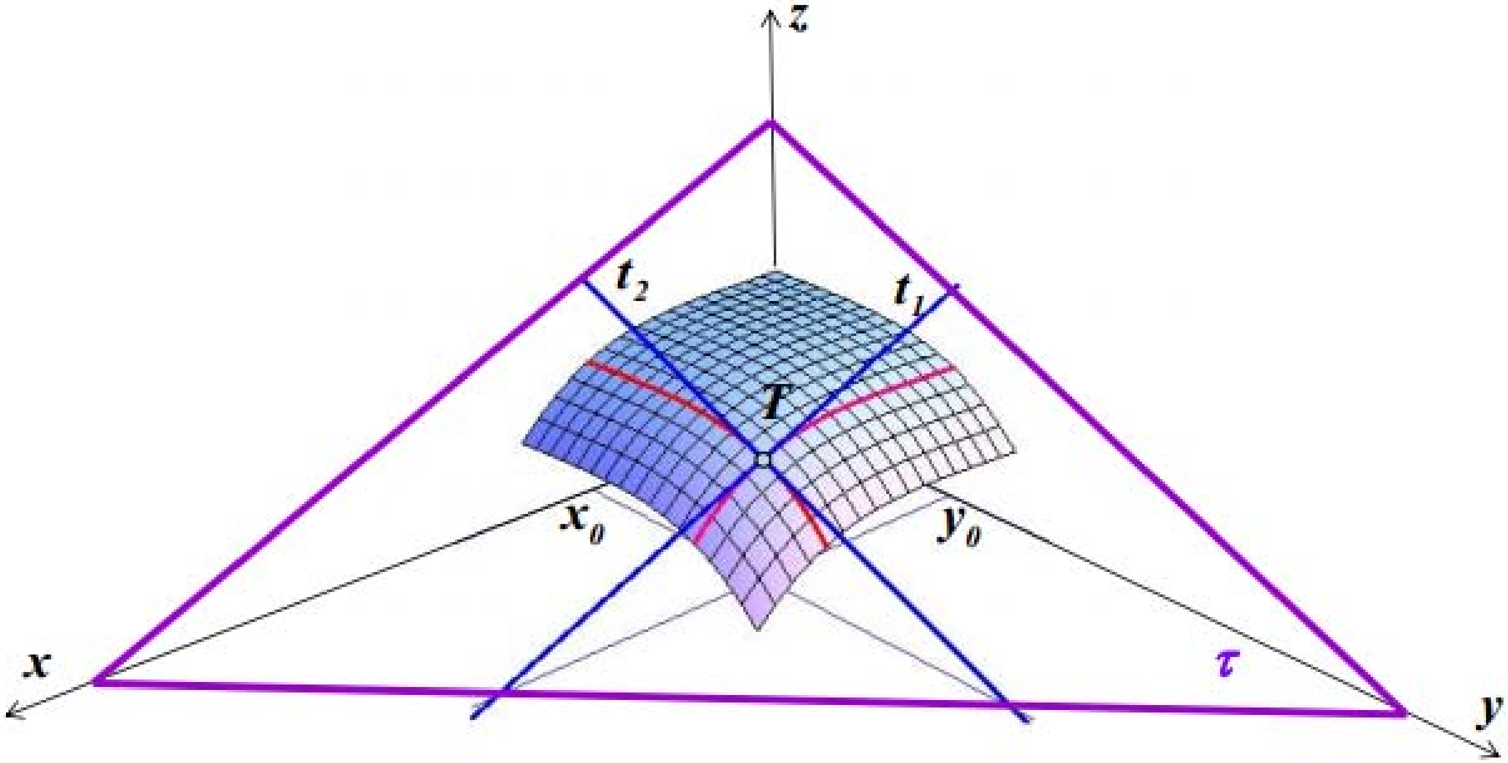
$$T=\left[x_0, y_0, f\left(x_0, y_0\right)\right]$$



$$tg\beta = f'_y\left(x_0, y_0\right) \qquad \vec{s}_{t_2}=\left(0, 1, f'_y\left(x_0, y_0\right)\right)$$

τ = (t_1, t_2) – is tangent plane to the graph of function $f(x, y)$ in the point

$$T=\left[x_0, y_0, f\left(x_0, y_0\right)\right]\in G\left(f\right)$$



Normal vector of plane τ - $\vec{\mathbf{n}} = \vec{\mathbf{s}}_{t_1} \times \vec{\mathbf{s}}_{t_2} = (-f'_x(x_0, y_0), -f'_y(x_0, y_0), 1)$

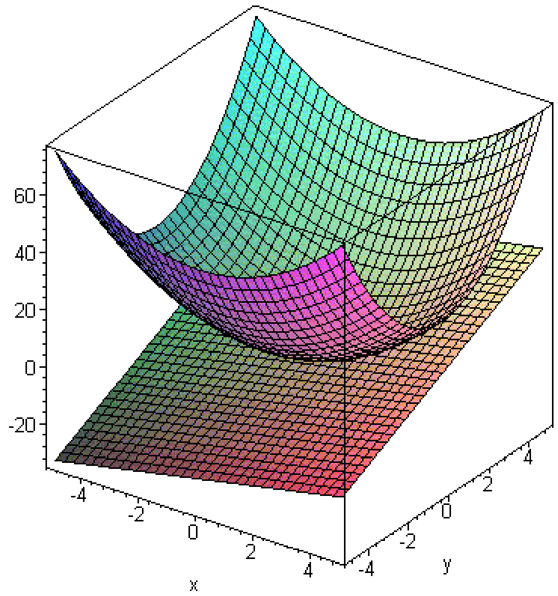
Let $A = [x_0, y_0]$ be limit point of domain of definition $D(f)$ of function $f(x, y)$ and let there exist both partial derivatives of function f in A

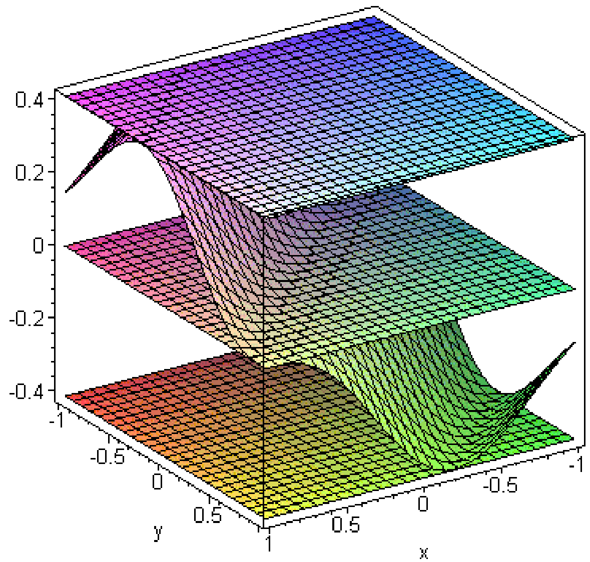
$$f'_x(x_0, y_0) = f'_x(A),\ f'_y(x_0, y_0) = f'_y(A)$$

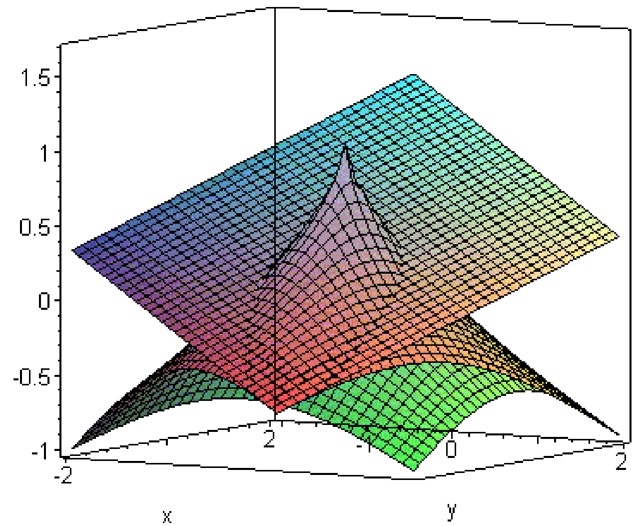
Tangent plane τ to the graph of function f in the tanget point

$T = [x_0, y_0, f(x_0, y_0)]$ is determined by equation

$$z - f(A) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0)$$


60
40
20
0
-20
-4
-2
0
2
4
x
-4
-2
0
2
4
y


0.4
0.2
0
-0.2
-0.4
-1
-0.5
0
0.5
1
y
0.5
0
-0.5
-1
x


1.5
1
0.5
0
-0.5
-1
-2
x
2
-1
0
2
y

Let $A = [x_0, y_0]$ be the limit point of domain $D(f)$ of definition of function $f(x, y)$ and let both partial derivatives f'_x, f'_y of function f exist and be continuous in point A.
Then function f is said to be differentiable in the point A.
Formula

$$df_A(x,y) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0)$$

is the total differential of function f in the point A.

If:

1. $\lim\limits_{X \to A} d(X, A) = 0$
2. **function f is continuous in the point A**
3. **partial derivatives $f'_x(A), f'_y(A)$ exist**

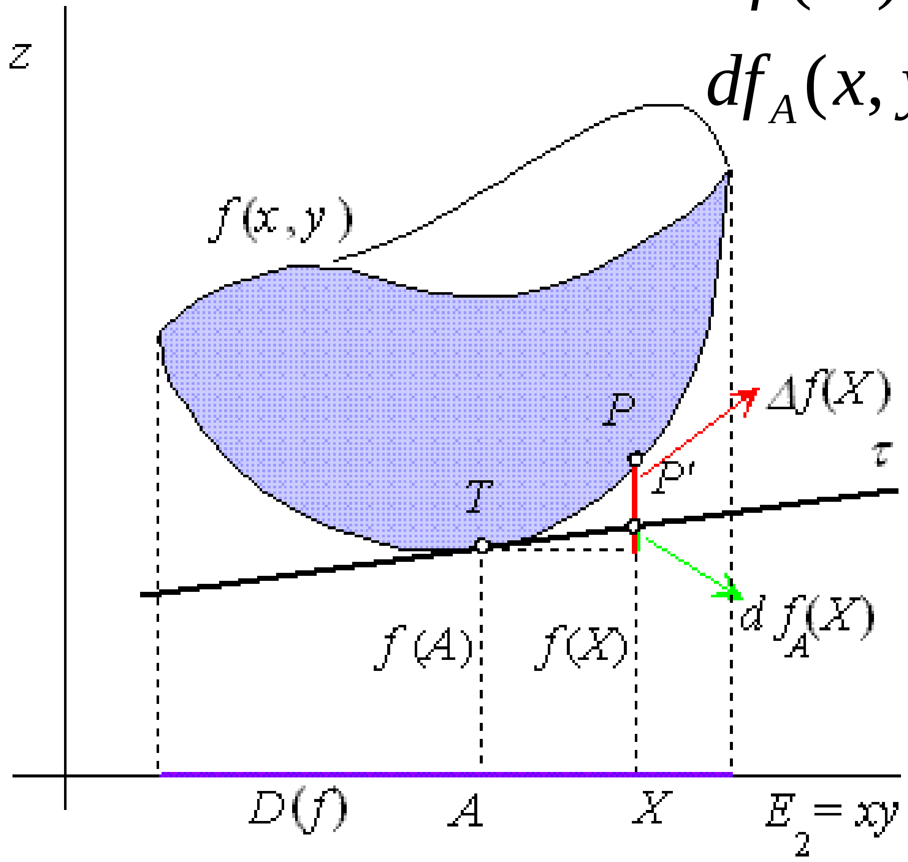
then

$$f(X) - f(A) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0)$$

Geometric interpretation of function differential

$$\Delta f(X) = f(X) - f(A)$$

$$df_A(x,y) = f'_x(A)(x-x_0) + f'_y(A)(y-y_0)$$



$$\lim_{X\to A} d(X-A) = 0 \Rightarrow df_A(X) \doteq \Delta f(X)$$