

Functions in more variables

Let M be non-empty subset of the n -dimensional Euclidean space

$$M \subset \mathbf{E}_n, n \geq 1, M \neq \emptyset.$$

Any mapping f from the set M to $\mathbf{R}$, in which a point $X=[x_1, x_2, \dots, x_n]$ of the space $\mathbf{E}_n$ is attached a unique real number, is called a real function of n real variables.

$$f: M \rightarrow \mathbf{R}, X \rightarrow f(X)$$

Set M is called domain of definition of function f - denoted as $D(f)$.

Image of the point $X = [x_1, x_2, \dots, x_n] \in M$ in the mapping f , the real number attached to the point X , denoted as $y = f(X) = f(x_1, x_2, \dots, x_n)$ is value of the function f in the point X .

Range of values of function f is the set

$$H(f) = \{y \in \mathbf{R}; \exists X \in D(f), y = f(X)\}$$

For $n = 2$ we use denotation $f(x, y)$ instead of $f(x_1, x_2)$

For $n = 3$ we use denotation $f(x, y, z)$ instead of $f(x_1, x_2, x_3)$.

Function of n variables is uniquely determined by its domain of definition $D(f) \subset E_n$ and a formula (or function rule), due to which exactly one real number y can be attached to an arbitrary point $X = [x_1, x_2, \dots, x_n] \in D(f)$ as a function value $y = f(X)$.

In the same way as for functions of one variable, function rule can be determined in various forms

- in words

- by table of values

- by graph

- analytically – by means of a mathematical expression or equation.

Let f be a function in n variables defined on the set M of points in the space E_n , $n \geq 1$. **Graph** of function f is a set $G(f)$ consisting of all such points $[x_1, x_2, \dots, x_n, x_{n+1}] \in E_{n+1}$, for which the following hold:

1. $[x_1, x_2, \dots, x_n] \in M$
2. $x_{n+1} = f(x_1, x_2, \dots, x_n)$

Geometrically we can directly visualise only graphs of functions in one or two variables.

Graph of function of one real variable $y = f(x)$ is a curve in the plane.

Graph of function in two variables $f(x,y)$ is defined on the set $D(f)$ of points in the space E_2 and it is the set of all such points $[x, y, z]$ in the space E_3 , for which:

1. $[x, y] \in D(f)$

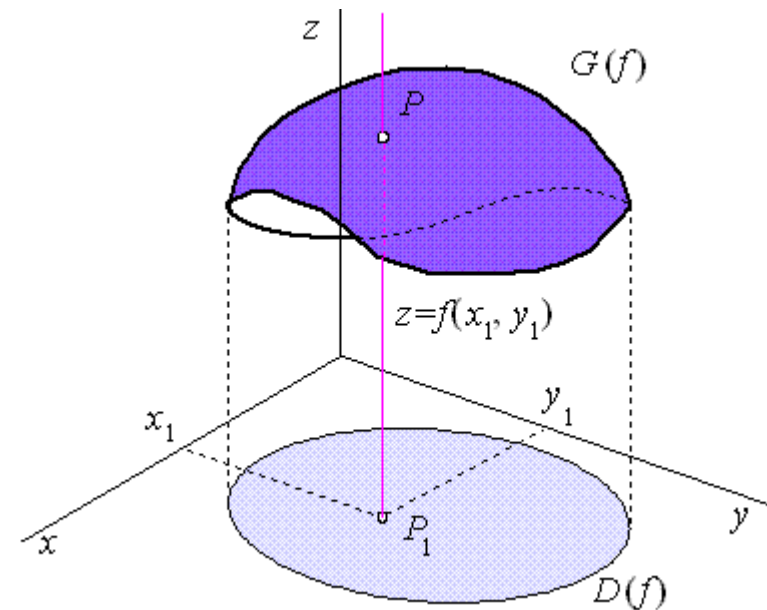
2. $z = f(x, y)$

$$G(f) = \{[x, y, z] \in E_3: [x, y] \in D(f), z = f(x, y)\}$$

Graph of function in two variables $f(x, y)$ is a set of those points $[x, y, z]$ of the space, whose coordinates satisfy equation

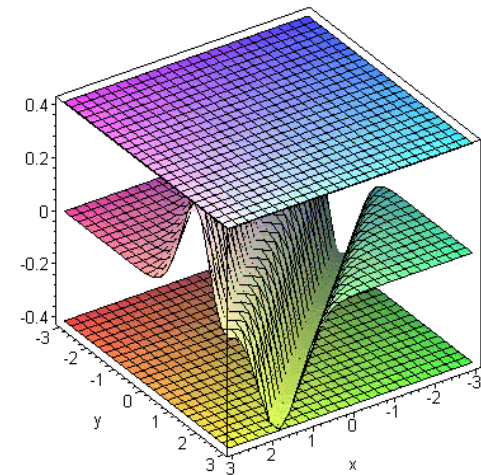
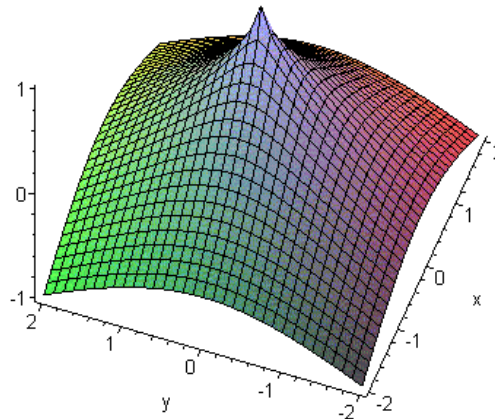
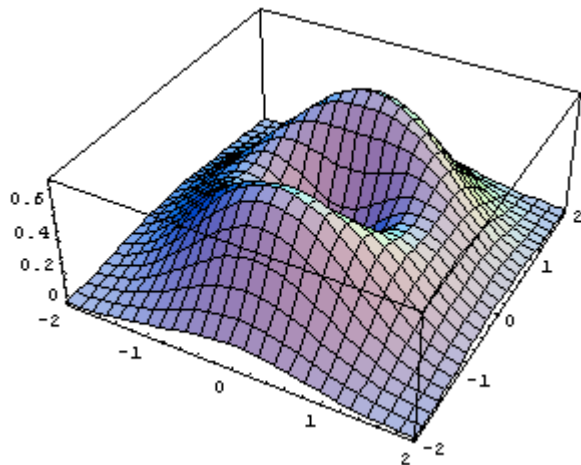
$$z = f(x, y)$$

i.e. it is a surface $G(f)$ in the space E_3 .



Properties:

1. Orthogonal projection of the graph of function $f(x, y)$ to the plane xy is the domain of definition $D(f)$ of function $f(x, y)$.
2. Any line parallel to the coordinate axis z intersects graph of the function $f(x, y)$ in one point at most.



Graph of function $f(x, y)$ can be projected in views by methods of Descriptive geometry by one of the projection methods – Monge method (top view and front view), axonometry (axonometric view).

Limit and continuity of functions in more variables

Let function $f(X)$ be defined on some neighbourhood of the point $A = [a_1, a_2, \dots, a_n]$, which is the limit point of the function domain of definition $D(f)$.

Number b is said to be limit of function $f(X)$ in the point A , if for all $\varepsilon > 0$ there exists such $\delta > 0$, that for all points $X \in O_\delta(A)$, $X \neq A$, holds $f(X) \in O_\varepsilon(b)$.

$$\lim_{X \rightarrow A} f(X) = b \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : d(X, A) < \delta \Rightarrow |f(X) - b| < \varepsilon$$

For function $f(X)$ in two variables and point $A = [x_0, y_0]$, we can write also

$$\lim_{X \rightarrow [x_0, y_0]} f(X) = b$$

Let functions f and g have in the point A limits

$$\lim_{X \rightarrow A} f(X) = b_1, \lim_{X \rightarrow B} g(X) = b_2$$

Then there exists in the point A also limit of the function:

1. $c_1 f + c_2 g$, where c_1, c_2 are arbitrary constants and

$$\lim_{X \rightarrow A} (c_1 f(X) + c_2 g(X)) = c_1 b_1 + c_2 b_2$$

2. $f \cdot g$ and

$$\lim_{X \rightarrow A} f(X) \cdot g(X) = b_1 b_2$$

3. $\frac{f}{g}$ and

$$\lim_{X \rightarrow A} \frac{f(X)}{g(X)} = \frac{b_1}{b_2}, b_2 \neq 0$$

Improper limit of function in more variables in the proper point, which is the limit point of its domain of definition, is defined similarly, as improper limit of function of one variable.

$$\lim_{X \rightarrow A} f(X) = \infty \Leftrightarrow \forall K > 0 \exists \delta > 0 : d(X, A) < \delta \Rightarrow f(X) > K$$

$$\lim_{X \rightarrow A} f(X) = -\infty \Leftrightarrow \forall K > 0 \exists \delta > 0 : d(X, A) < \delta \Rightarrow f(X) < -K$$

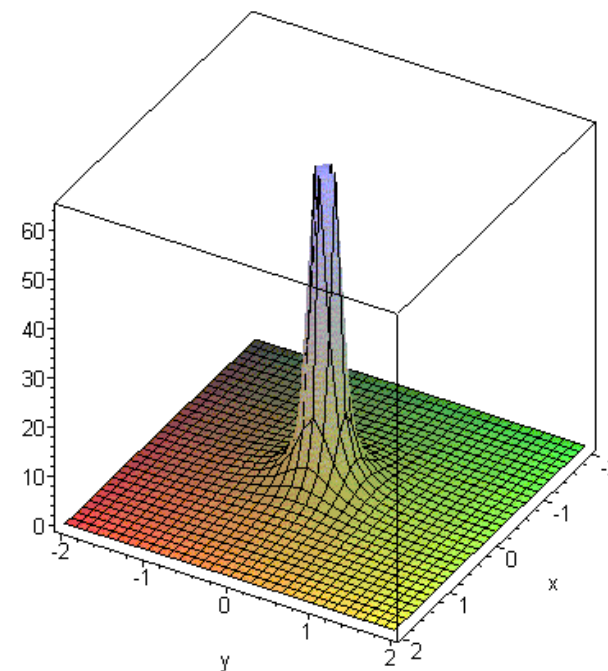
Function $f(x, y) = \frac{1}{x^2 + y^2}$

is not defined in the point $[0, 0]$, which is the limit point of domain $D(f) = \mathbf{E}_2 - \{[0, 0]\}$

It holds $\lim_{X \rightarrow [0,0]} (x^2 + y^2) = 0$

therefore there exists improper limit

$$\lim_{X \rightarrow [0,0]} f(x, y) = \lim_{X \rightarrow [0,0]} \frac{1}{x^2 + y^2} = \infty$$



Let function $f(X)$ be defined on some neighbourhood of the point $A = [a_1, a_2, \dots, a_n]$, which is the limit point of the function domain of definition $D(f)$. Function $f(X)$ is continuous in the point A , if there exists proper limit of function f in the point A , and this equals to the function f value in the point A .

$$\lim_{X \rightarrow A} f(X) = f(A)$$

Function f continuous in all points of the set $M \subset D(f)$, is said to be continuous on the set M .

If $M = D(f)$, we speak about continuous function f .

Let functions f and g be defined on some neighbourhood of the point A and they are both continuous in the point A , then also functions

$$c_1 f + c_2 g, \quad c_1, c_2 \in \mathbf{R} \quad \text{and} \quad f \cdot g,$$

are continuous in the point A .

If $g(A) \neq 0$, then also function $\frac{f}{g}$ is continuous in the point A .

Function in more variables continuous on a closed region has similar properties as function in one variable continuous on a closed interval.

Let function f be continuous on a bounded, connected and closed region $\Omega \subset E_n$, then:

1. Function f is bounded on Ω , therefore there exists such positive number $K > 0$, that $|f(X)| < K$ for any point $X \in \Omega$.
2. Function f has its maximum and minimum on the set Ω , therefore there exists at least one point $P_1 \in \Omega$ and at least one point $P_2 \in \Omega$ such, that $f(X) \leq f(P_1)$, and $f(X) \geq f(P_2)$ for any point $X \in \Omega$.
3. Let A, B be different points from region Ω , such that $f(A) \neq f(B)$. Then function f reaches all values between $f(A)$ and $f(B)$ at least in one point from the region Ω , therefore there exists at least one point $C \in \Omega$ such, that

$$f(A) < f(C) < f(B).$$

Range of values of function f in two variables, which is continuous on bounded closed region Ω , which is the image of region Ω in the mapping determined by function f , is a closed interval in $\mathbf{R}$.

