Double integrals

Let f be function with two variables defined on region $M \subset \mathrm{E}_2$

$M = \{[x, y] \in \mathrm{E}_2 : a \le x \le b, c \le y \le d\} =$

$= \langle a, b\rangle \times \langle c, d\rangle$ - 2D interval

$f(x, y) \ge 0$

graph is a surface patch $G(f)$ in E_3 over M.

Solid in E_3 is bounded by planes

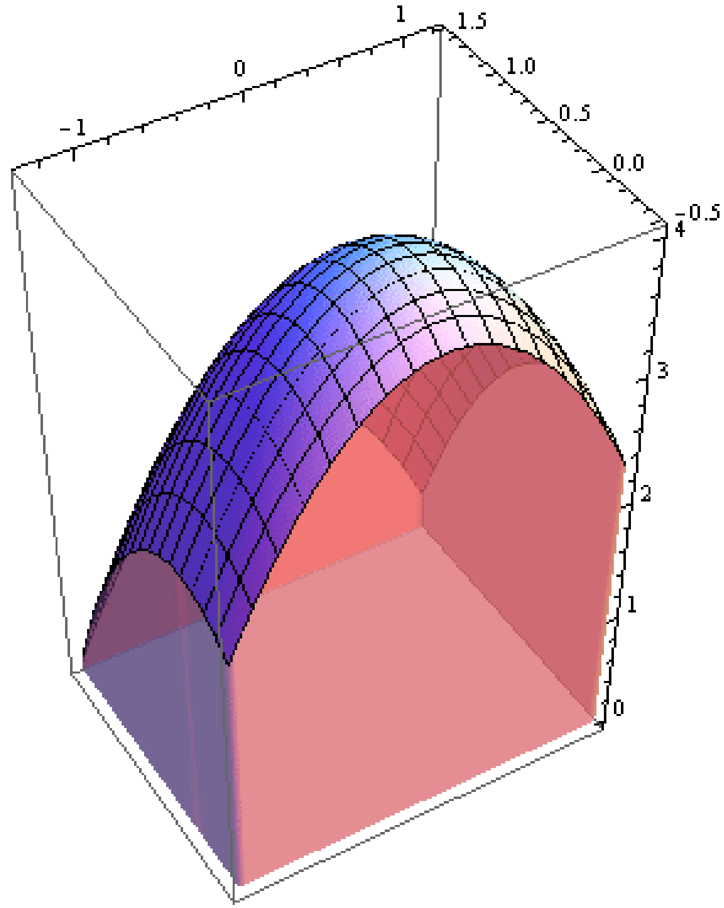
$$z = 0, x = a, x = b, \mathrm{y} = c, y = d$$

and surface patch with equation $\mathrm{z} = f(x, y)$.

$T = \{[x, y, z] \in \mathrm{E}_3 : [x, y] \in M, 0 \le z \le f(x, y)\}$

How can be calculated the volume of this solid

$V = ?$



division of interval $\langle a, b\rangle$ $\quad a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

division of interval $\langle c, d\rangle$ - $\quad c = y_0 < y_1 < \dots < y_{m-1} < y_m = d$

division of region M $\quad M_{ij} = \langle x_{i-1}, x_i\rangle \times \langle y_{j-1}, y_j\rangle,\ i = 1,\dots, n,\ j = 1,\dots, m$

division of solid T onto $k = m.nl$ partial „prisms”

Taking arbitrary points $[\xi_i, \eta_j] \in M_{ij}$ function f value is $f(\xi_i, \eta_j)$

and we can compute sum of the values $f(\xi_i, \eta_j).P(M_{ij})$,

for area of partial rectangular $P(M_{ij}) = \Delta x_i \Delta y_j$

Integral sum of function $f(x, y)$ on a rectangular region M is

$$\sum_{i,j=1}^{n,m} f(\xi_i, \eta_j).P(M_{ij})$$

If there exists for an arbitrary choice of points $[\xi_i, \eta_j] \in M_{ij}$ the same proper limit

$$\lim_{\max P(M_{ij}) \to 0} \sum_{i,j=1}^{m,n} f(\xi_i, \eta_j).P(M_{ij})$$

then this is called double integral of function f on region M

$$\iint_M f(x,y)\,dxdy$$

Function f is said to be integrable on region M.

Geometric interpretation of double integral of function $f(x, y) \geq 0$ on region M is the volume of solid T.

Sufficient condition for integrability

Let function *f* with two variables be bounded on region *M*

and let there exist a finite number of points of discontinuity,

then function *f* is integrable on *M*.

Consequence

Any function *$f(x, y)$* that is continuous on region *M* is integrable

on this region.

FUBINI theorem

Let function $f(x, y)$ be continuous on region $M = \langle a, b\rangle \times \langle c, d\rangle$, then

$$\iint_M f(x, y)\,dx\,dy =$$

$$= \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left(\int_a^b f(x, y)\,dx \right) dy =$$

$$= \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx$$

Properties of double integrals

1. **Linearity**

   Let functions f_1, f_2, ..., f_k be integrable on region M and c_1, c_2, ..., c_k are real numbers, then

$$\iint_M \left(c_1 f_1(x,y) + c_2 f_2(x,y) + \ldots + c_k f_k(x,y)\right) dxdy =$$

$$= c_1 \iint_M f_1(x,y)dxdy + c_2 \iint_M f_2(x,y)dxdy + \ldots + c_k \iint_M f_k(x,y)dxdy$$

2. **Aditivity**

   Let function f be integrable on region M that is a union of a finite number of measurable regions M_i with no common interior points. Then

$$M = \bigcup_{i=1}^{k} M_i \qquad \iint_M f(x,y)dxdy = \sum_{i=1}^{k} \iint_{M_i} f(x,y)dxdy$$

3. **Monotonicity**

   Let functions *f*, *g* be integrable on region *M* and let $\forall X = [x, y] \in M$ holds $f(x, y) \leq g(x, y)$, then

$$\iint_M f(x,y)\,dxdy \leq \iint_M g(x,y)\,dxdy$$

4. **Let function *f* be integrable on a measurable region *M* and $f(x, y) \geqq 0$ for all $X = [x, y] \in M$, then**

$$\iint_M f(x,y)\,dxdy \geq 0$$

5. **Let function *f* be integrable on a measurable region *M*, then also function $|f|$ is integrable on *M* and**

$$\left|\iint_M f(x,y)\,dxdy\right| \leq \iint_M |f(x,y)|\,dxdy$$

Set $M = \{[x, y] \in E_2 \colon a \le x \le b, g(x) \le y \le h(x)\}$

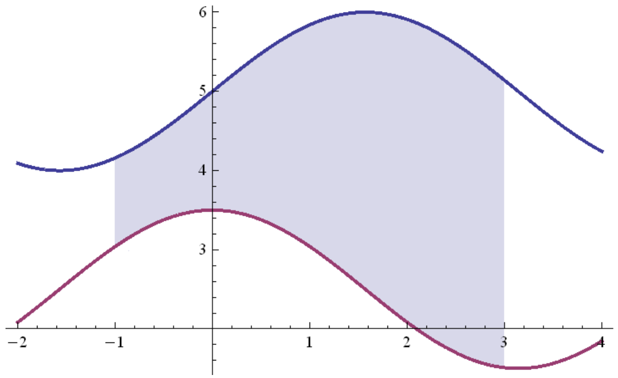
for $a, b \in R, a < b$,

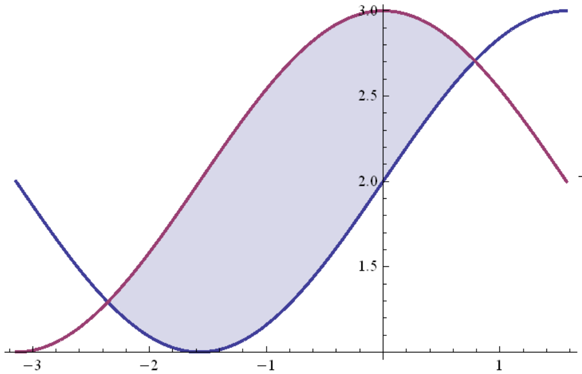
Continuous functions g and h defined on

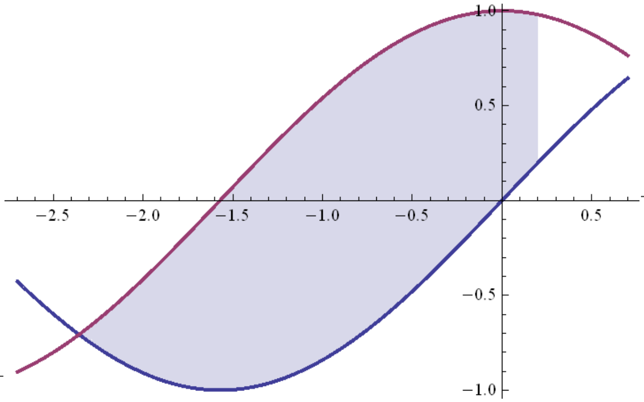
interval $\langle a, b\rangle$, while for all $x \in \langle a, b\rangle$ holds

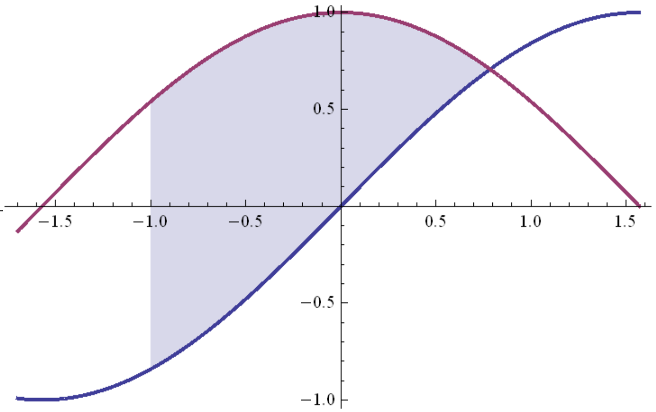
$g(x) \le h(x)$,

is called regular region of type x









Set $M = \{[x, y] \in E_2 : g(y) \le x \le h(y),\ c \le y \le d\}$, for $c, d \in R$, $c < d$, functions g and h continuous on interval $\langle c, d\rangle$, while for all $y \in \langle c, d\rangle$ holds $g(y) \le h(y)$, is called regular region of type y.

Double integral of function f with two variables on a regular region M can be defined similarly as on region $\langle a, b\rangle \times \langle c, d\rangle$.

Let function f be integrable on regular region M, then double integral exists

$$\iint_M f(x, y)\,dxdy$$

All properties of double integral on region $\langle a, b\rangle \times \langle c, d\rangle$ are valid analogously also for double integral on regular region *M*, or on set, which is a union of finite number of elementary regions.

1- **Linearity**

2- **Aditivity**

3- **Monotonicity**

4- **Positivity**

5- $$\left| \iint_M f(x, y)\,dxdy \right| \leq \iint_M |f(x, y)|\,dxdy$$

FUBINI theorem

Let function $f(x, y)$ be continuous on regular region

$M = \{[x, y] \in E_2: a \le x \le b, g(x) \le y \le h(x)\}$ of type x, then

$$\iint_M f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

For function $f(x, y)$ continuous on regular region

$M = \{[x, y] \in E_2: g(y) \le x \le h(y), c \le y \le d\}$ holds

$$\iint_M f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

Triple integrals

Let function $f(x, y, z)$ with three variables be defined on a three-dimensional region $M \subset E_3$

$$M = \{[x, y, z] \in E_3 : a \leqq x \leqq b, c \leqq y \leqq d, e \leqq z \leqq h\}$$

$$M = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle$$

with volume $V(M) = (b - a)(d - c)(h - e)$

Dividing intervals $\langle a, b\rangle, \langle c, d\rangle, \langle e, h\rangle$ and applying similar steps as for definition of double integral of functions with two variables, the integral sum of function $f(x, y, z)$ on set M can be determined

$$\sum_{i,j,k=1}^{l,m,n} f(\xi_i, \eta_j, \zeta_k).V(M_{ijk})$$

Let function *f* be defined on a bounded region *M*. If for an arbitrary choice of points $[\xi_i, \eta_j, \zeta_k] \in M_{ijk}$ there exists a proper limit

$$\lim_{\max V(M_{ijk}) \to 0} \sum_{i,j,k=1}^{l,m,n} f(\xi_i, \eta_j, \zeta_k).V(M_{ijk})$$

it is called the triple integral of function *f* on region *M* and denoted

$$\iiint_M f(x,y,z)\,dxdydz$$

Function *f* is said to be integrable on region *M*.
Sufficient condition for integrability of function *f* is to be bounded on set *M*. Any function continuous on region *M* is integrable on *M*.

FUBINI theorem

For any function *f*(*x*, *y*, *z*) that is continuous on region

***M* = ⟨*a*, *b*⟩ × ⟨*c*, *d*⟩ × ⟨*e*, *h*⟩ holds**

$$\iiint_M f(x,y,z)\,dx\,dy\,dz = \int_e^h \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz =$$

$$= \int_e^h \left(\int_c^d \int_a^b f(x,y,z)\,dx\,dy \right) dz = \int_e^h \left(\int_a^b \int_c^d f(x,y,z)\,dy\,dx \right) dz =$$

$$\int_a^b \left(\int_c^d \int_e^h f(x,y,z)\,dz\,dy \right) dx = \int_c^d \left(\int_e^h \int_a^b f(x,y,z)\,dx\,dz \right) dy$$

Regular region with respect to plane *xy*

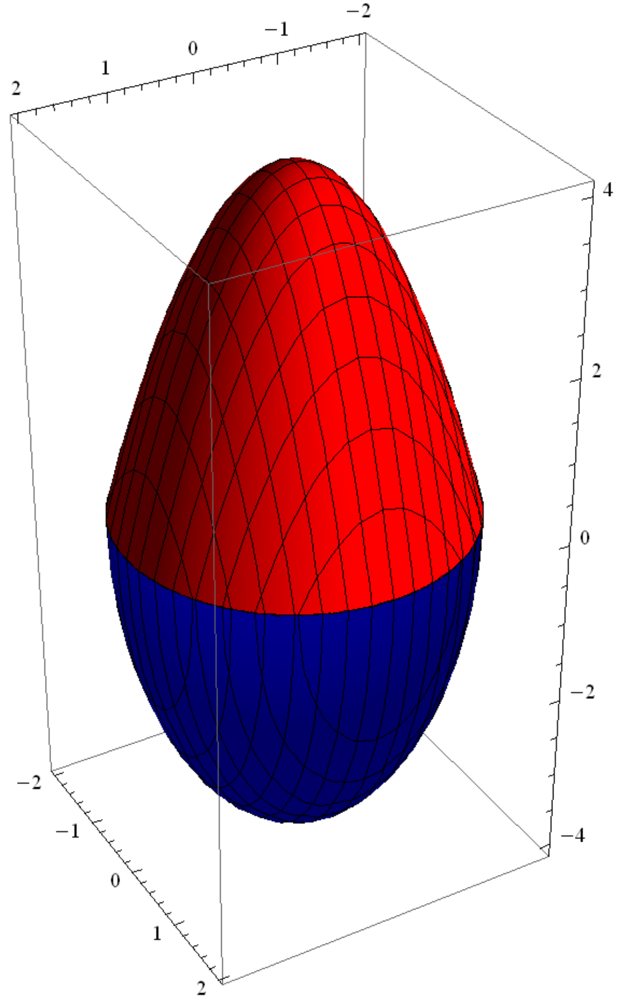
Let functions $z_1(x, y) \leq z_2(x, y)$

be continuous and bounded on regular region

$M = \{[x, y] \in E_2: a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

or

$M' = \{[x, y] \in E_2: h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$



Set of points

$D = \{[x, y, z] \in E_3: [x, y] \in M, z_1(x, y) \leq z \leq z_2(x, y)\}$

$D' = \{[x, y, z] \in E_3: [x, y] \in M', z_1(x, y) \leq z \leq z_2(x, y)\}$

is a regular region in E_3 with respect to the plane *xy*.

FUBINI theorem

For function *f*(*x*, *y*, *z*) continuous on a regular region *D* holds

$$\iiint_D f(x,y,z)\,dxdydz = \iint_M \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dzdxdy =$$

$$= \int_a^b \left\{ \int_{g_1(x)}^{g_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dy \right\} dx$$

If function *f*(*x*, *y*, *z*) is continuous on an regular region *D*′, then

$$\iiint\limits_{D'} f(x,y,z)\,dxdydz = \iint\limits_{M'} \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dzdxdy =$$

$$= \int\limits_{c}^{d} \left\{ \int\limits_{h_1(y)}^{h_2(y)} \left[\int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dx \right\} dy$$

All properties of double integral on region $\langle a, b\rangle \times \langle c, d\rangle$, or on a regular region M, analogously hold also for a triple integral on region $\langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle$, or on a regular region D, or on set, which is a union of a finite number of regular regions.

1- **Linearity**

2- **Aditivity**

3- **Monotonicity**

4- **Positivity**

5- $$\left| \iiint_D f(x, y, z) dxdydz \right| \leq \iiint_D \left| f(x, y, z) \right| dxdydz$$