Scalar and vector fields

Let $\Omega \subset \boldsymbol{E}^3$ be a space region – open or closed connected subset of space $\boldsymbol{E}^3$ determined by Cartesian coordinates, ordered triples of real numbers

$X = [x, y, z] \in \boldsymbol{R}^3$.

Let f be a scalar function, such that any point $X \in \Omega$ is attached a real number. Ordered pair (Ω, f) is called a stationary scalar field, while function f is called the potential of this field.

$$f(X) = f(x, y, z) = h,\ h \in H(f) \subset \boldsymbol{R}$$

To illustrate „graph" of function f we would need 4-dimensional space $\boldsymbol{E}^4$.

Let function $f(X)$ be continuous on region Ω and let it have continuous partial derivatives with respect to all variables, which are not simultaneously all equal to zero. Set of points in the region Ω, at which the potencial has the same value $C \in H(f)$, forms a surface in the space $\boldsymbol{E}^3$ with equation satisfied by coordinates of points: $f(x, y, z) = C$.

Surfaces determined by equations

$$f(x, y, z) = C_i\,,\; C_{\mathrm{i}} \in H(f)$$

are called equipotential (level) surfaces of the scalar field (Ω, f).

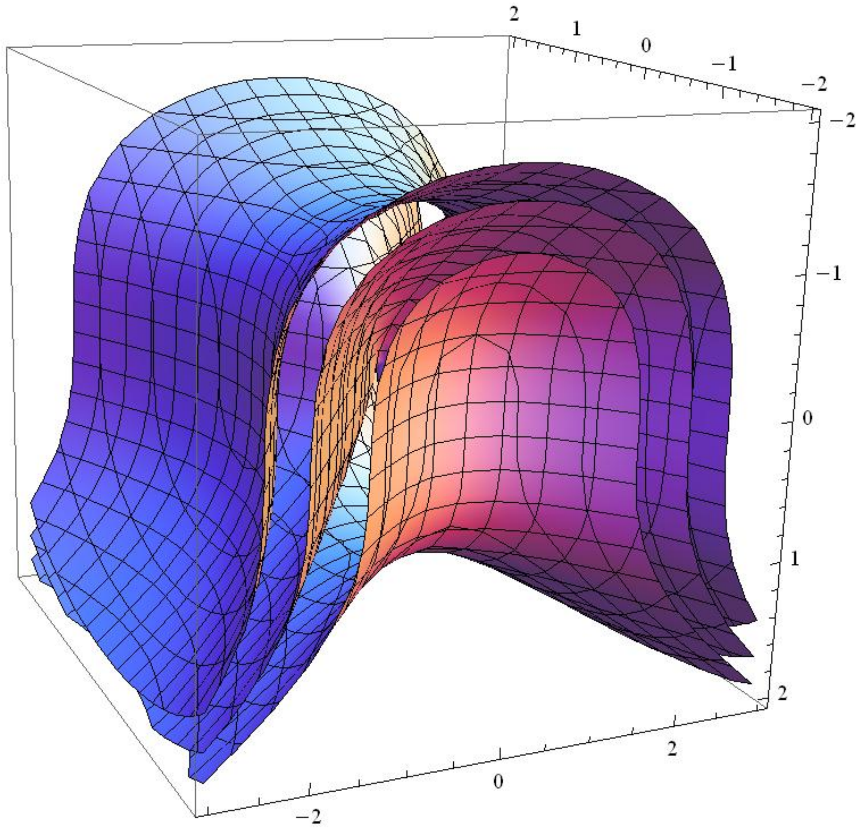
These are e.g. isotermic surfaces in the termostatic field, or isobaric surfaces in the field of the astmospheric pressure.

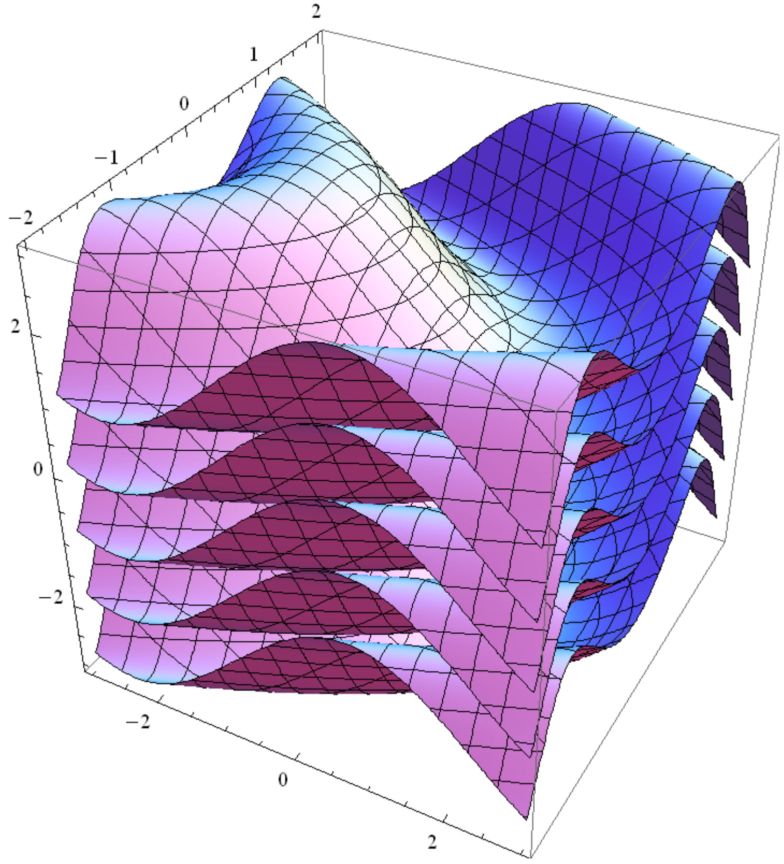
Equipotential surfaces of the electrostatic field generated by a given point power source q are concentric spheres with centre at the point q.

System of all equipotential surfaces of a scalar field (Ω, f) corresponding to all values of potential fill in the entire region Ω.

Exactly one equipotential surface is passing through each point in the region, and no two equipotential surfaces for $C_i \neq C_j$ have a common point.

Space scalar field, graph of function of three variables, three-dimensional manifold (solid) in $\boldsymbol{E}^4$, can be visualised by equipotential surfaces in $\boldsymbol{E}^3$





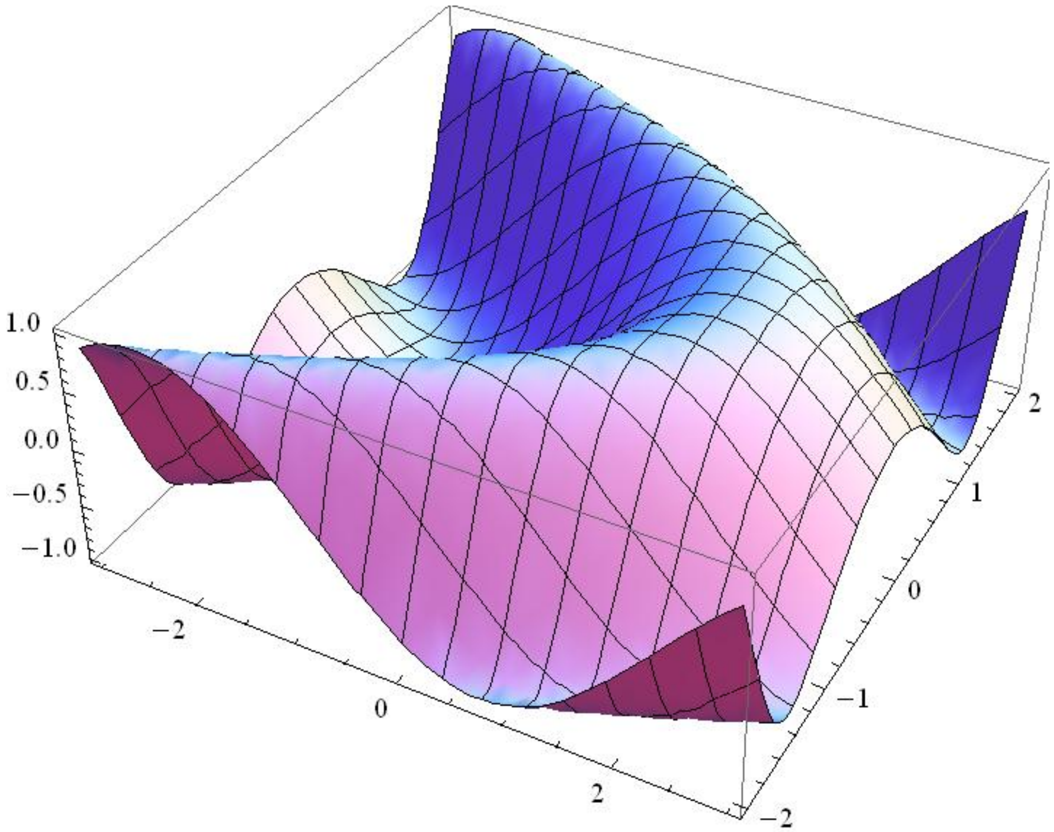
$F(x, y, z) = x^3 + y^2 - z^2$

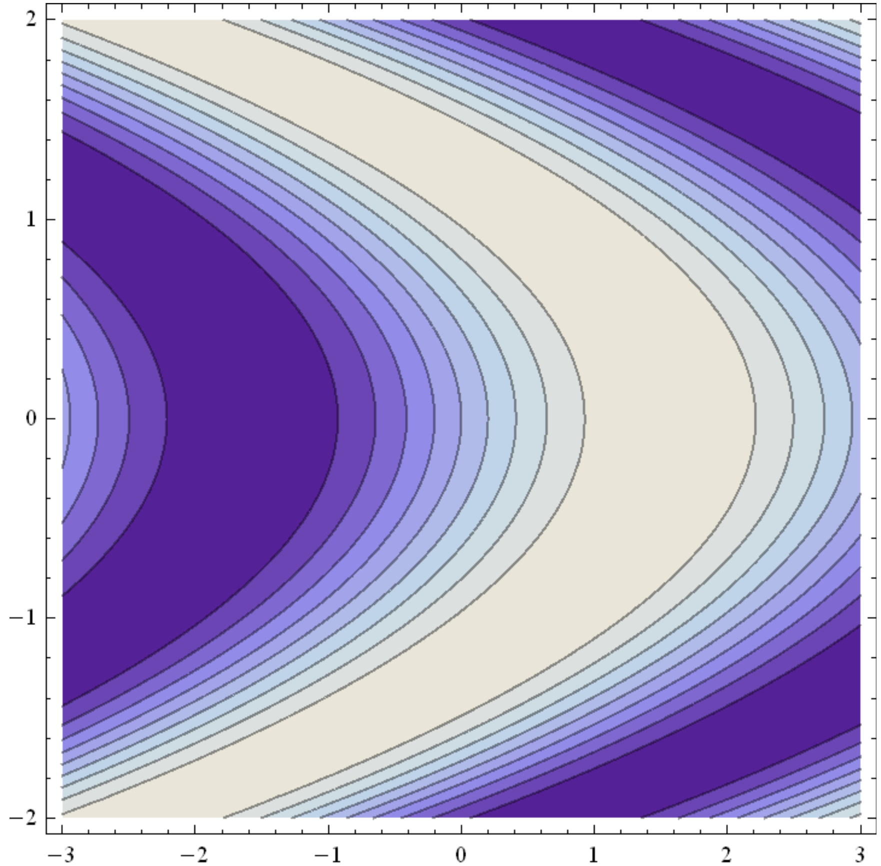
$F(x, y, z) = \sin(x + y^2\ ) + z$

Plane scalar field $U(x, y) = C$

Equipotential curves = level curves of the graph of function of two variables

$U(x, y) = \sin(x + y^2)$





Derivative in a given direction – directional derivative

Function $f(x, y, z)$ differentiable at the point X_0 has at this point derivative in any direction **s** and the following holds

$$f'_{\mathbf{s}}(X_0) = f'_x(X_0)\cos\alpha + f'_y(X_0)\cos\beta + f'_z(X_0)\cos\gamma$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of vector **s**.

If $\mathbf{s} = \mathbf{i}$, then $\alpha = 0$, $\beta = \pi/2$, a $\gamma = \pi/2$, $f'_{\mathbf{i}}(X_0) = f'_x(X_0)$
If $\mathbf{s} = \mathbf{j}$, then $\alpha = \pi/2$, $\beta = 0$, a $\gamma = \pi/2$, $f'_{\mathbf{j}}(X_0) = f'_y(X_0)$
If $\mathbf{s} = \mathbf{k}$, then $\alpha = \pi/2$, $\beta = \pi/2$, a $\gamma = 0$, $f'_{\mathbf{k}}(X_0) = f'_z(X_0)$

Partial derivatives of function f at the point X_0 with respect to variables x, y, z are derivatives of function f at the point X_0 in direction of unit vectors **i**, **j**, **k** that are direction vectors of coordinate axes x, y and z.
Derivative of function $f(x, y, z)$ in direction **s** is scalar product of unit vector

$\mathbf{s}_0 = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$ and vector $f'_x(X_0)\mathbf{i} + f'_y(X_0)\mathbf{j} + f'_z(X_0)\mathbf{k}$

$$f'_{\mathbf{s}}(X_0) = (f'_x(X_0), f'_y(X_0), f'_z(X_0)).(\cos\alpha, \cos\beta, \cos\gamma)$$

Gradient of a scalar function $f(x, y, z)$ at the point X_0

$$\operatorname{grad} f(X_0) = f'_x(X_0)\mathbf{i} + f'_y(X_0)\mathbf{j} + f'_z(X_0)\mathbf{k}$$

is such vector, in direction of which the derivative of function f at the point X_0 is maximal and equals to the norm of this vector $[f'_{\mathbf{s}}(X_0)]_{\max} = |\operatorname{grad} f(X_0)|$

$$f'_{\mathbf{s}}(X_0) = |\operatorname{grad} f(X_0)| \cos \varphi$$

where φ is angle formed by vectors $\operatorname{grad} f(X_0)$ and $\mathbf{s}$.

If $\operatorname{grad} f(X_0) = \mathbf{0}$, derivative of function f at the point X_0 in any direction $\mathbf{s}$ vanishes, $f'_{\mathbf{s}}(X_0) = 0$ for any vector $\mathbf{s}$.

Physical meaning of a gradient at a point

Values of function $f(x, y, z)$ are changing most rapidly (are increasing) in the direction of vector that is the function gradient, i.e. in direction

$$\operatorname{grad} f(X_0) = (f'_x(X_0), f'_y(X_0), f'_z(X_0)).$$

Geometric meaning of a gradient at a point

Let function f be differentiable at the point X_0 and let grad $f(X_0) \neq \boldsymbol{0}$,

$$[f'_x(X_0)]^2 + [f'_y(X_0)]^2 + [f'_z(X_0)]^2 \neq 0$$

then a surface determined by equation

$$f'_x(x_0)(x - x_0) + f'_y(x_0)(y - y_0) + f'_z(x_0)(z - z_0) = 0$$

is tangent to the equipotential surface of the scalar field (Ω, f), passing

through the point $X_0 = [x_0, y_0, z_0]$.

Vector $$\text{grad } f(x_0, y_0, z_0) = f'_x(X_0)\mathbf{i} + f'_y(X_0)\mathbf{j} + f'_z(X_0)\mathbf{k}$$

is a normal vector to the tangent plane to the equipotential surface at the given tangent point $X_0 = [x_0, y_0, z_0]$.

Derivative of function f in direction $\mathbf{s}$ at the point X_0 equals to the norm of an orthographic projection of vector grad $f(X_0)$ to the vector $\mathbf{s}$.

Let (Ω, f) be a scalar field, and let function f be differentiable at all points $X \in \Omega$. Vector function $\mathbf{F}$ = grad f defined on region Ω , by which any point $X \in \Omega$ is attached a vector grad $f(X)$ is called gradient of function f

$$\mathbf{F} = \text{grad}\, f = f'_x\,\mathbf{i} + f'_y\,\mathbf{j} + f'_z\,\boldsymbol{k}$$

or

$$\mathbf{F} = \text{grad}\ f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Vector function $\mathbf{F}$ = grad f is called gradient of the scalar field (Ω, f)

$$\mathbf{F} = \text{grad}\ f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

To any scalar field (Ω, f) correspondes a unique vector field

$(\Omega, \mathbf{F})$, where $\mathbf{F}$ = grad f, which is the vector field of gradients of a scalar field (Ω, f).

Gradient of a scalar field is a differential characteristcs of this scalar field.

Properties of gradient of a scalar function

1. $\operatorname{grad}(f_1 + f_2 + ... + f_n) = \operatorname{grad} f_1 + \operatorname{grad} f_2 + ... + \operatorname{grad} f_n$

2. $\operatorname{grad}(f\,g) = f \operatorname{grad}(g) + g \operatorname{grad}(f)$ $\qquad$ $\operatorname{grad}(f(g)) = f'(g) \operatorname{grad}(g)$

3. $$\left|\operatorname{grad} f\right| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}$$

Symbolic vector $\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$ is called the **Hamiltonian differential operator ∇ = nabla**

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Derivative of function in direction **s** equals

$$f'_{\mathbf{s}}(X) = \nabla f \cdot \mathbf{s}_0, \mathbf{s}_0 = \frac{\mathbf{s}}{|\mathbf{s}|}$$

Let **F** be a vector function, by which any point $X = (x, y, z) \in \Omega$ is attached a vector in the space $\boldsymbol{E}^3$

$$\mathbf{F}(X) = \mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$$

An ordered pair $(\Omega, \mathbf{F})$ is said to be a stationary vector field.
Hodograph of function **F** is a manifold in the 3-dimensional space $\boldsymbol{E}^3$.

A good view into the vector field can be given by vector curves.

Vector curve k (flow curve - flow) of the vector field $(\Omega, \mathbf{F})$ is a regular curve in the region Ω determined parametrically as

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ t \in \boldsymbol{R}$$

and its tangent vector $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

at the point $X(t) = [x(t), y(t), z(t)]$, $t \in \boldsymbol{R}$ is collinear to vector $\mathbf{F}(X(t))$, as

$$\frac{x'(t)}{f_1(x,y,z)} = \frac{y'(t)}{f_2(x,y,z)} = \frac{z'(t)}{f_3(x,y,z)}$$

Equations

$$\frac{x'(t)}{f_1(x,y,z)} = \frac{y'(t)}{f_2(x,y,z)}, \quad \frac{x'(t)}{f_1(x,y,z)} = \frac{z'(t)}{f_3(x,y,z)}, \quad \frac{y'(t)}{f_2(x,y,z)} = \frac{z'(t)}{f_3(x,y,z)}$$

represent a system of non-linear differential equation of order 1.

Solution of this system is a triple of functions $x(t), y(t), z(t)$, $t \in \boldsymbol{R}$ determining system of vector curves of the respective vector field.

Exactly one vector curve of the vector field $(\Omega, \boldsymbol{F})$ is passing through any point $X_0 \in \Omega$, at which $\mathbf{F}(X_0) \neq \mathbf{0}$.

Differential characteristics of a vector field

Divergence of a vector field (vector function) *F* at the point is scalar

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Vector field is said to be solenoidal, if div **F** = 0.

Field with non-zero divergence is rotational field, in which there exists at least one flow, therefore

$$\operatorname{div}\mathbf{F}(X) \neq 0 \text{ at at least one point } X = [x, y, z] \in \Omega .$$

Properties of divergence

div (**F** + **G**) = div **F** + div **G**

div (f **F**) = f div **F** + **F** . grad f

Curl of a vector field (vector function F) at the point is a vector

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\times(f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix} =$$

$$= \left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z},\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x},\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Properties: curl(**F** + **G**) = curl **F** + curl **G**, curl (f **F**) = f curl **F** – **F** × grad f

A vector field for which the curl vanishes is said to be an irrotational field, or conservative field.

Rotational field contains whirls at those points, at which curl is non-zero, and it determines the direction of a flow at this point.

Vector field, which is a gradient of a scalar field $f(x, y, z)$ is irrotational, and any irrotational field can be represented as a gradient of a scalar field.

Laplace operator Δ

Scalar product of nabla opertor ∇ with itself is the Laplace operator (Laplacian) Δ

$$\Delta = \nabla.\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{, if } \Delta f = 0, \text{ function is said to be harmonic}$$

$$\Delta \mathbf{F} = (\Delta f_1, \Delta f_2, \Delta f_3) = (\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}, \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}, \frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2})$$

Properties of Laplace operator

$\Delta(f + g) = \Delta f + \Delta g, \quad \Delta(f\,g) = f\,\Delta g + g\,\Delta f + 2\nabla f\,.\nabla g$

$\Delta\,\nabla f = \nabla\,(\Delta f)$

$\Delta(\mathbf{F} + \mathbf{G}) = \Delta\mathbf{F} + \Delta\mathbf{G}, \qquad \Delta\,\text{curl}\,\mathbf{F} = \text{curl}\,\Delta\mathbf{F}$

Formulas for calculations with differential operators

$\nabla(f\,g)$ = grad $(f\,g)$ = f grad g + g grad f vector

$\nabla(f\,\mathbf{F})$ = div $(f\,\mathbf{F})$ = f div $\mathbf{F}$ + $\mathbf{F}$ grad f scalar

$\nabla \times (f\,\mathbf{F})$ = curl $(f\,\mathbf{F})$ = f curl $\mathbf{F}$ – $\mathbf{F} \times$ grad f vector

$\nabla(\mathbf{F.G})$ = grad $(\mathbf{F.G})$ = $\mathbf{F} \times$ curl $\mathbf{G}$ + $\mathbf{G} \times$ curl $\mathbf{F}$ +

$$+\left(f_1\frac{\partial\mathbf{G}}{\partial x}+f_2\frac{\partial\mathbf{G}}{\partial y}+f_3\frac{\partial\mathbf{G}}{\partial z}\right)+\left(g_1\frac{\partial\mathbf{F}}{\partial x}+g_2\frac{\partial\mathbf{F}}{\partial y}+g_3\frac{\partial\mathbf{F}}{\partial z}\right)\ \text{vector}$$

$\nabla\,(\mathbf{F} \times \mathbf{G})$ = div $(\mathbf{F} \times \mathbf{G})$ = $\mathbf{G}$ curl $\mathbf{F}$ – $\mathbf{F} \times$ curl $\mathbf{G}$ scalar

$\nabla \times (\mathbf{F} \times \mathbf{G})$ = curl $(\mathbf{F} \times \mathbf{G})$ = $\mathbf{F}$ div $\mathbf{G}$ – $\mathbf{G}$ div $\mathbf{F}$ +

$$+\left(g_1\frac{\partial\mathbf{F}}{\partial x}+g_2\frac{\partial\mathbf{F}}{\partial y}+g_3\frac{\partial\mathbf{F}}{\partial z}\right)-\left(f_1\frac{\partial\mathbf{G}}{\partial x}+f_2\frac{\partial\mathbf{G}}{\partial y}+f_3\frac{\partial\mathbf{G}}{\partial z}\right)\ \text{vector}$$

$\nabla^2 f = \nabla\nabla f = \text{div grad } f = \text{div } \nabla\, f = \Delta f$	scalar
$\nabla \times (\nabla f) = \text{curl grad } f = \text{curl } \nabla\, f = \mathbf{0}$	vector
$\nabla(\nabla \mathbf{F}) = \text{grad div } \mathbf{F} = \text{curl curl } \mathbf{F} + \Delta\, \mathbf{F}$	vector
$\nabla(\nabla \times \mathbf{F}) = \text{div curl } \mathbf{F} = 0$	scalar

Basic relations

$\nabla f = \text{grad } f$	vector (direction of increase-decrease of function values)
$\nabla.\mathbf{F} = \text{div } \mathbf{F}$	scalar (existence of flows)
$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$	vector (direction and magnitude of whirls)
$\nabla^2 f = \Delta f$	scalar (divergence of gradient, gradient change, curvature of the scalar field)
$\nabla^2 \mathbf{F} = \Delta\, \mathbf{F}$	vector (direction and magnitude of curvature of vector field)