

Surfaces in the space E^3

Let vector function in two variables be defined on region $\Omega \subset \mathbf{R}^2$

$$\mathbf{p}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in \Omega,$$

whose scalar coordinate functions $x(u, v), y(u, v), z(u, v)$ are at least once differentiable on region Ω .

Hodograph of vector function $\mathbf{p}(u, v)$ is a piece-wise smooth surface χ in $\mathbf{E}^3$.

For any ordered pair of real numbers $(u_0, v_0) \in \Omega$ the function value is position vector $\mathbf{p}(u_0, v_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ of surface point $P(u_0, v_0)$, with Cartesian coordinates

$$P(u_0, v_0) = [x(u_0, v_0), y(u_0, v_0), z(u_0, v_0)].$$

Ordered pair of numbers $(u_0, v_0) \in \Omega$ is called curvilinear coordinates of point on surface.

If $\Omega \subset \mathbf{R}^2$ is a regular region, with boundary in a closed regular curve in $\mathbf{E}^2$, then surface is called elementary surface or surface patch.

All points on surface determined by curvilinear coordinates

$$(u, v_0) \in \Omega, v_0 = \text{const.}, \text{ resp. } (u_0, v) \in \Omega, u_0 = \text{const.},$$

are point on one curve on the surface, which is called

iso-parametric u -curve - $\mathbf{p}(u, v_0) = (x(u, v_0), y(u, v_0), z(u, v_0))$

or iso-parametric v -curve - $\mathbf{p}(u_0, v) = (x(u_0, v), y(u_0, v), z(u_0, v))$.

Iso-parametric curves form two systems of curves on surface, they form curvilinear net of curves on surface called iso-parametric net of curves.

Any curve from one system of curves intersects all curves in the other system of curves on the surface.

One iso-parametric curve from each of the two systems is passing through any surface point, while it holds

$$P(u_0, v_0) = \mathbf{p}(u, v_0) \cap \mathbf{p}(u_0, v)$$

Point $P(u_0, v_0)$ on the surface is called regular point, if there exist continuous first partial derivatives $\mathbf{p}_u, \mathbf{p}_v$ of vector function $\mathbf{p}(u, v)$ on some neighbourhood of $(u, v_0) \in \Omega$, and it holds

$$\mathbf{p}'_u(u_0, v_0) \times \mathbf{p}'_v(u_0, v_0) \neq \mathbf{0}$$

In the opposite case, when vectors $\mathbf{p}'_u(u_0, v_0), \mathbf{p}'_v(u_0, v_0)$ are linearly dependent, we speak about singular point on surface.

Vector functions

$$\mathbf{p}'_u(u, v_0) = (x'_u(u, v_0), y'_u(u, v_0), z'_u(u, v_0))$$

$$\mathbf{p}'_v(u_0, v) = (x'_v(u_0, v), y'_v(u_0, v), z'_v(u_0, v))$$

define direction vector fields of tangents to iso-parametric curves on surface passing through the surface regular point $P(u_0, v_0)$.

At any regular point $P(u_0, v_0)$ there exist 2 linearly independent vectors
 tangent vector to iso-parametric u -curve
 and tangent vector to iso-parametric v -curve,

$$\mathbf{p}'_u(u_0, v_0) = (x'_u(u_0, v_0), y'_u(u_0, v_0), z'_u(u_0, v_0))$$

$$\mathbf{p}'_v(u_0, v_0) = (x'_v(u_0, v_0), y'_v(u_0, v_0), z'_v(u_0, v_0))$$

that determine a unique tangent plane at the given point

$$\tau = (P, \mathbf{p}'_u, \mathbf{p}'_v)$$

Equation of tangent plane $[\mathbf{P} - \mathbf{p}, \mathbf{p}'_u, \mathbf{p}'_v] = 0$

$$\begin{vmatrix} X - x(u_0, v_0) & Y - y(u_0, v_0) & Z - z(u_0, v_0) \\ x'_u(u_0, v_0) & y'_u(u_0, v_0) & z'_u(u_0, v_0) \\ x'_v(u_0, v_0) & y'_v(u_0, v_0) & z'_v(u_0, v_0) \end{vmatrix} = 0$$

Vector $\mathbf{n}(u_0, v_0) = \mathbf{p}'_u(u_0, v_0) \times \mathbf{p}'_v(u_0, v_0) \neq \mathbf{0}$

is called normal vector to surface at the regular point $P(u_0, v_0)$.

Line determined by direction vector $\mathbf{n}$ and passing through point $P(u_0, v_0)$ is a normal line to surface that is perpendicular to the tangent plane τ to surface in this point.

Vector function $\mathbf{n}(u, v) = \frac{\mathbf{p}'_u(u, v) \times \mathbf{p}'_v(u, v)}{|\mathbf{p}'_u(u, v) \times \mathbf{p}'_v(u, v)|}$

defines direction vector field of normals to surface in regular points and determines orientation of surface. Value of function $\mathbf{n}(u, v)$ is a unit vector, whose coordinates are direction cosines of normals to surface at the regular points, while

$$\mathbf{n} = \cos\alpha \mathbf{i} + \cos\beta \mathbf{j} + \cos\gamma \mathbf{k}$$

where α, β, γ are angles that vector $\mathbf{n}$ forms to unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Let $\mathbf{p}(u(t), v(t))$, $t \in I$ is curve on surface $\mathbf{p}(u, v)$, $(u, v) \in \Omega \subset R^2$.
Tangent vector to the curve can be represented by differential

$$d\mathbf{p} = (\mathbf{p}'_u u' + \mathbf{p}'_v v') dt$$

Square of this differential satisfies equation

$$|d\mathbf{p}|^2 = \mathbf{p}'_u \mathbf{p}'_u du^2 + 2\mathbf{p}'_u \mathbf{p}'_v du dv + \mathbf{p}'_v \mathbf{p}'_v dv^2$$

Denoting

$$E = \mathbf{p}'_u \mathbf{p}'_u, F = \mathbf{p}'_u \mathbf{p}'_v, G = \mathbf{p}'_v \mathbf{p}'_v$$

we can receive form

$$E du^2 + 2F du dv + G dv^2$$

that is called **the first fundamental differential form of surface**,
or the first tensor of surface, denoted also as φ_1 .

The first differential form of surface φ_1 describes interior properties of surface geometry – lengths of curve segments and their angles.

Formula $D_1 = EG - F^2$ is called **discriminant of the first differential form**.

The first differential form of surface φ_1 is positively definite at any regular point on surface, and it holds:

1. $E = |\mathbf{p}_u|^2 > 0, G = |\mathbf{p}_v|^2 > 0$
2. $EG - F^2 = |\mathbf{p}_u|^2 |\mathbf{p}_v|^2 - |\mathbf{p}_u \mathbf{p}_v|^2 = |\mathbf{p}_u \times \mathbf{p}_v|^2 > 0$
3. symmetric means that scalar product is commutative

Length of curve segment on surface

Let $\mathbf{p}(u(t), v(t)), t \in \langle a, b \rangle$ is a curve on surface $\mathbf{p}(u, v), (u, v) \in \Omega \subset R^2$, whose first differential form is $ds^2 = Edu^2 + 2Fdu dv + Gdv^2$

For the length $s(A, B)$ of segment, where $A = \mathbf{p}(u(a), v(a)), B = \mathbf{p}(u(b), v(b))$ holds

$$s(A, B) = \int_a^b \sqrt{ds^2} = \int_a^b \sqrt{E(t)(u'(t))^2 + 2F(t)u'(t)v'(t) + G(t)(v'(t))^2} dt$$

Angle of two curves on surface

Let curves k_1 and k_2 be determined by their parameterizations $u_1(t), v_1(t)$ and $u_2(t), v_2(t)$, $t \in \langle a, b \rangle$ on surface $\mathbf{p}(u, v)$, $(u, v) \in \Omega \subset R^2$ and let them intersect in the surface point $P(u_0, v_0)$, whose curvilinear coordinate on the curve k_1 is t_1 and on the curve k_2 it is t_2 .

Angle ω of these curves can be calculated as angle of their tangent vectors $\mathbf{p}_1, \mathbf{p}_2$, and the following holds

$$\cos \omega = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1| |\mathbf{p}_2|} = \frac{Eu'_1(t_1)u'_2(t_2) + 2F(u'_1(t_1)v'_2(t_2) + u'_2(t_2)v'_1(t_1)) + Gv'_1(t_1)v'_2(t_2)}{\sqrt{E(u'_1(t_1))^2 + 2Fu'_1(t_1)v'_1(t_1) + G(v'_1(t_1))^2} \sqrt{E(u'_2(t_2))^2 + 2Fu'_2(t_2)v'_2(t_2) + G(v'_2(t_2))^2}}$$

For the angle of iso-parametric curves at the point $P(u, v)$ it holds

$$\cos \omega = \frac{\mathbf{p}'_u \cdot \mathbf{p}'_v}{|\mathbf{p}'_u| |\mathbf{p}'_v|} = \frac{F}{\sqrt{EG}}$$

Area of elementary surface patch

Area element on a surface defined by vector function $\mathbf{p}(u, v)$ that is at least once differentiable on region $\Omega \subset R^2$ is parallelogram in the tangent plane to surface determined by vectors $\mathbf{p}_u du$ and $\mathbf{p}_v dv$ forming angle α

$$\begin{aligned} |dS| &= |\mathbf{p}'_u du| |\mathbf{p}'_v dv| \sin \alpha = |\mathbf{p}'_u du \times \mathbf{p}'_v dv| = |\mathbf{p}'_u \times \mathbf{p}'_v| |dudv| = \\ &= \sqrt{(\mathbf{p}'_u \times \mathbf{p}'_v)(\mathbf{p}'_u \times \mathbf{p}'_v)} |dudv| = \sqrt{EG - F^2} dudv \end{aligned}$$

Area of surface patch defined on region Ω can be calculated by means of double integral from the area element on the surface

$$\sigma(S) = \iint_{\Omega} |\mathbf{p}'_u(u, v) \times \mathbf{p}'_v(u, v)| dudv = \iint_{\Omega} \sqrt{D_1} dudv$$

The second fundamental differential form of a surface, or the second tensor of surface is related to the surface curvature in the regular point and it is denoted

$$\varphi_2 = \mathbf{n} \cdot d^2\mathbf{p} = Ldu^2 + 2Mdu \, dv + Ndv^2$$

while for its coefficients holds

$$L = \mathbf{n} \cdot \mathbf{p}_{uu}'' = \frac{(\mathbf{p}'_u \times \mathbf{p}'_v) \cdot \mathbf{p}_{uu}''}{|\mathbf{p}'_u \times \mathbf{p}'_v|} = \frac{[\mathbf{p}'_u, \mathbf{p}'_v, \mathbf{p}_{uu}'']}{\sqrt{D_1}}$$

$$M = \mathbf{n} \cdot \mathbf{p}_{uv}'' = \frac{(\mathbf{p}'_u \times \mathbf{p}'_v) \cdot \mathbf{p}_{uv}''}{|\mathbf{p}'_u \times \mathbf{p}'_v|} = \frac{[\mathbf{p}'_u, \mathbf{p}'_v, \mathbf{p}_{uv}'']}{\sqrt{D_1}}$$

$$N = \mathbf{n} \cdot \mathbf{p}_{vv}'' = \frac{(\mathbf{p}'_u \times \mathbf{p}'_v) \cdot \mathbf{p}_{vv}''}{|\mathbf{p}'_u \times \mathbf{p}'_v|} = \frac{[\mathbf{p}'_u, \mathbf{p}'_v, \mathbf{p}_{vv}'']}{\sqrt{D_1}}$$

Expression $D_2 = LN - M^2$ is called **discriminant of the second differential form**.

Number

$$K = \frac{LN - M^2}{EG - F^2} = \frac{D_2}{D_1}$$

is called total, or Gaussian curvature in the given point.

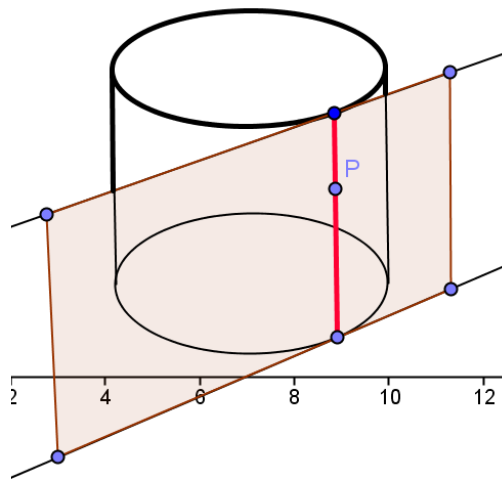
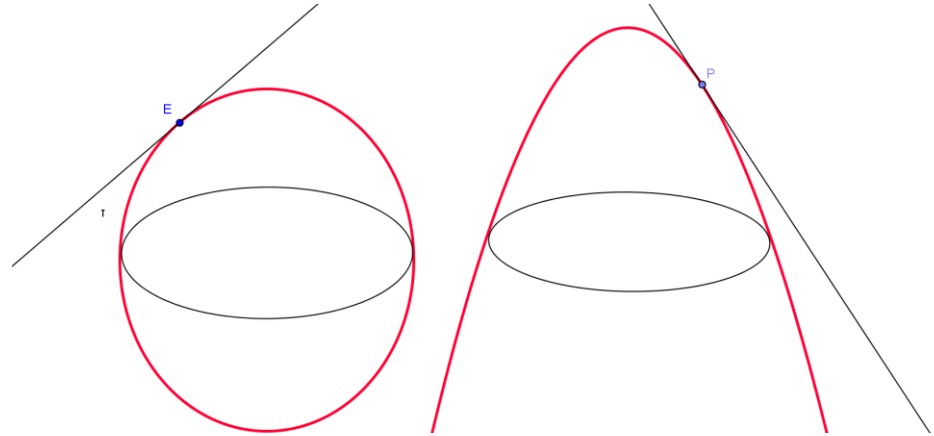
Surface point, at which $K > 0$, is called surface elliptic point. Tangent plane at this point has just one point in common with the surface, while surface is located in one half-space determined by tangent plane and normal vector to the surface at this point.

Surface point, at which $K < 0$, is called surface hyperbolic point. Tangent plane at this point intersects surface in a surface curve, which has a double point at the tangent point, while surface is located in both half-spaces determined by tangent plane at this point.

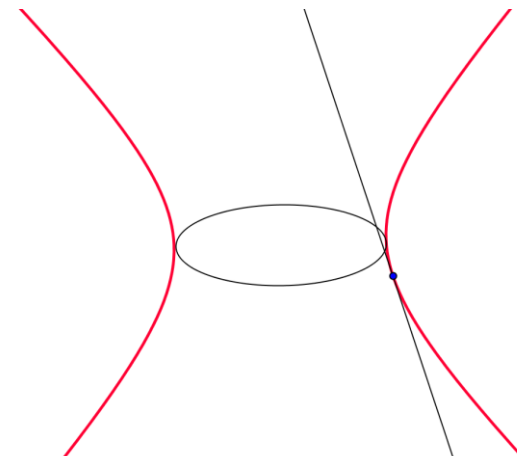
Surface point, at which $K = 0$, is called surface parabolic point. Tangent plane at this point is tangent to the surface in a curve, while surface is located in one half-space determined by tangent plane and normal vector to the surface at this point.

Theorema egregium – Gaussian curvature of the surface can be determined by means of coefficients of the first fundamental differential form of surface and their derivatives.

Surfaces with elliptic points
sphere, elipsoid, paraboloid



Surfaces with parabolic points
plane, cylindrical surface, conical surface



Surfaces with hyperbolic points
hyperboloid, pseudosphere

Surface with all types of points

torus

