

Curves in the space E^3

Smooth piece-wise continuous curve k in space E^3 is hodograph of a vector function $\mathbf{r}(t) = (x(t), y(t), z(t))$, defined on an interval in real numbers $I \subset \mathbf{R}$, while

1. all three scalar coordinate functions $x(t), y(t), z(t)$ are continuous on interval I
2. they have on I continuous derivatives $x'(t), y'(t), z'(t)$, from which at least one is non-zero.

Function values of scalar coordinate functions $x(t), y(t), z(t)$ at any real number $t_0 \in I$ define Cartesian coordinates of point on curve

$P(t_0) = [x(t_0), y(t_0), z(t_0)]$, as end point of vector

$\mathbf{r}(t_0) = (x(t_0), y(t_0), z(t_0))$

that is the value of function $\mathbf{r}(t)$ in the point t_0 .

Number $t_0 \in I$ is called curvilinear coordinate of point $P(t_0)$ on curve k .

For $I = \langle a, b \rangle$ we speak about curve segment with end points $P(a), P(b)$, while curve segment is oriented equally with the parametrization, therefore with the orientation of interval I .

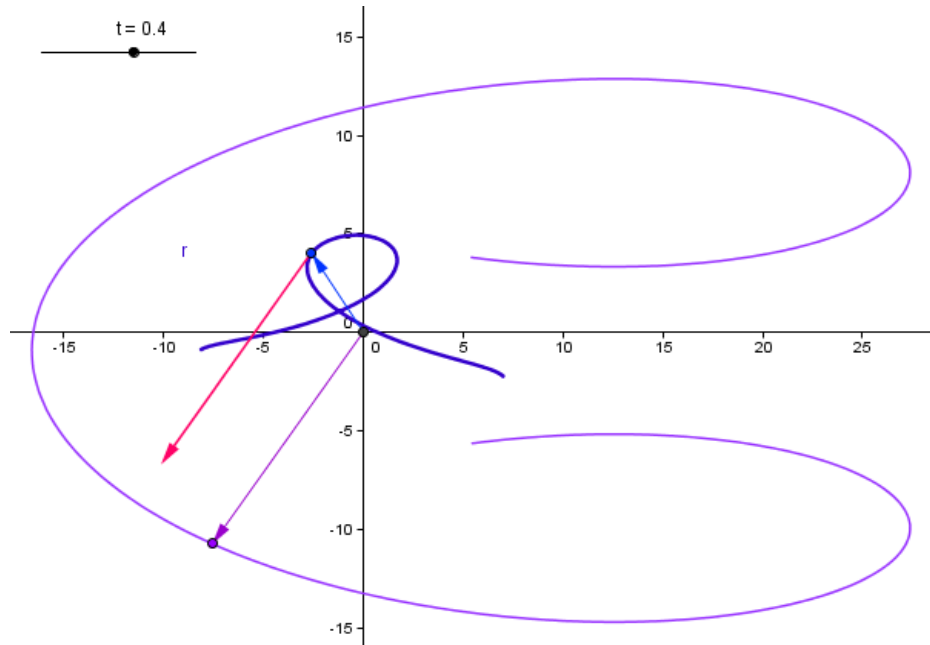
For $P(a) = P(b)$ we speak about closed curve.

Point on curve k is called m -multiple point, if there exist such values of variable $t_i \in I, i = 1, 2, \dots, m$, for which holds

$$\mathbf{r}(t_i) = (x(t_i), y(t_i), z(t_i)) = \mathbf{r}(t_j) = (x(t_j), y(t_j), z(t_j))$$

for all $i, j = 1, 2, \dots, m$

$P(a) = P(b)$ is a double point of the closed curve defined on interval $I = \langle a, b \rangle$.



Value of vector function $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ defined on interval $I \subset \mathbf{R}$ in the point $t_0 \in I$, vector

$$\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$

is called tangent vector to the curve k in the point $P(t_0)$.

Range of vector function $\mathbf{r}'(t)$ therefore defines direction vector field of tangents to curve k in its points.

Point $P(t_0)$ is called a regular point of the curve, if it holds

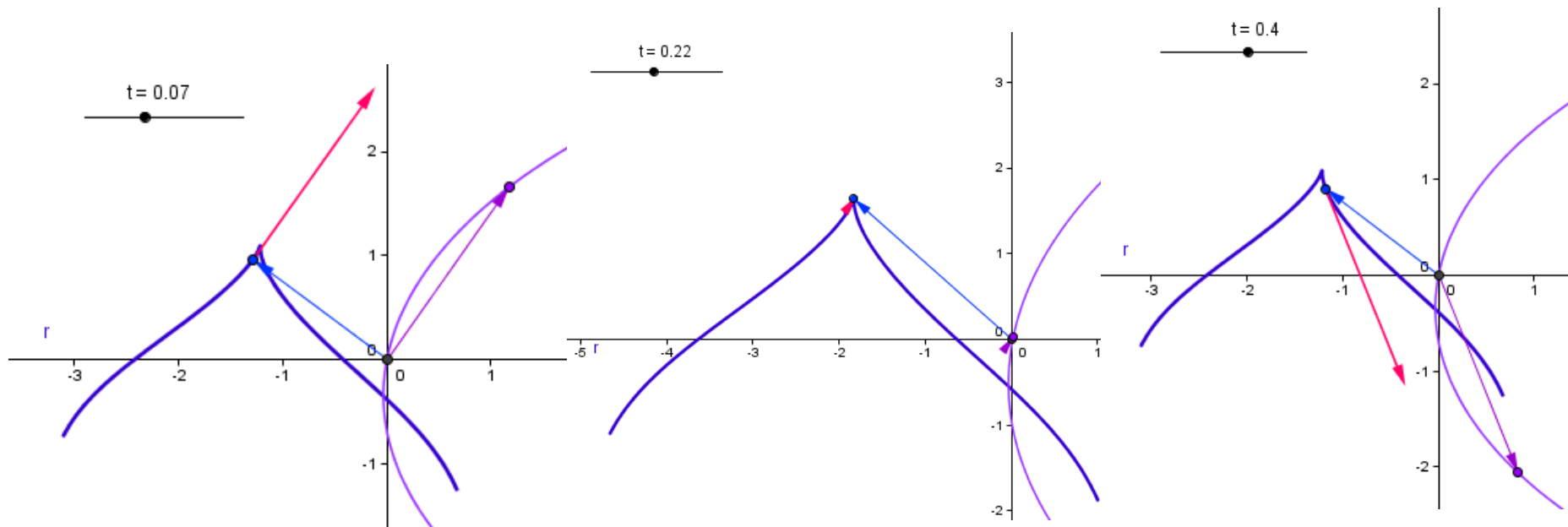
$$\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0)) \neq \mathbf{0}$$

In the opposite case we speak about singular point on curve k .

At any regular point $P(t_0)$ there exists tangent line to the curve, line passing through the given point with the direction vector $\mathbf{r}'(t_0)$.

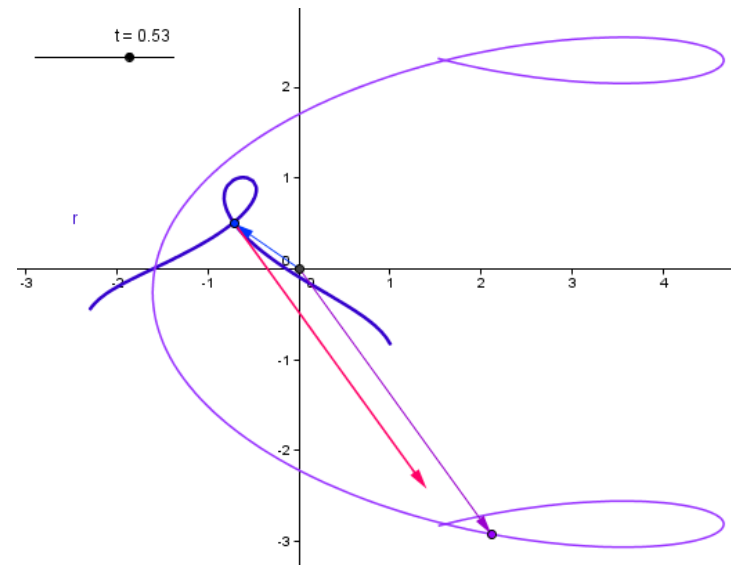
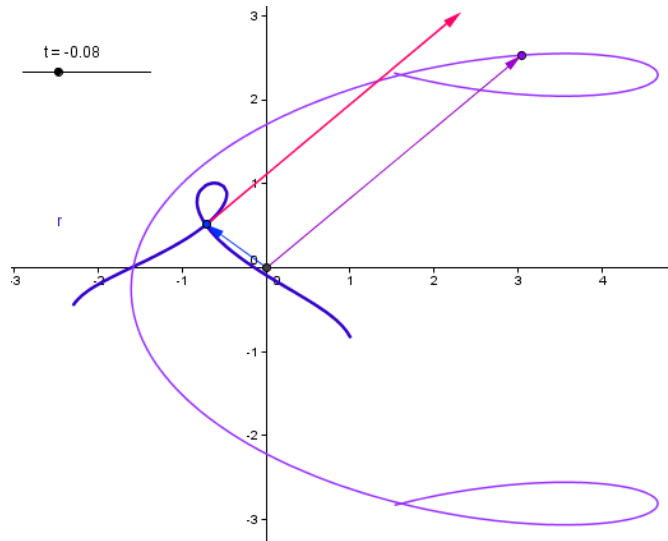
Orientation of vector $\mathbf{r}'(t_0)$ is equal to the orientation of the curve.

Point on curve, in which this orientation is changing, is called cuspidal point or cusp. It is a singular point on curve, in which $\mathbf{r}'(t) = \mathbf{0}$.



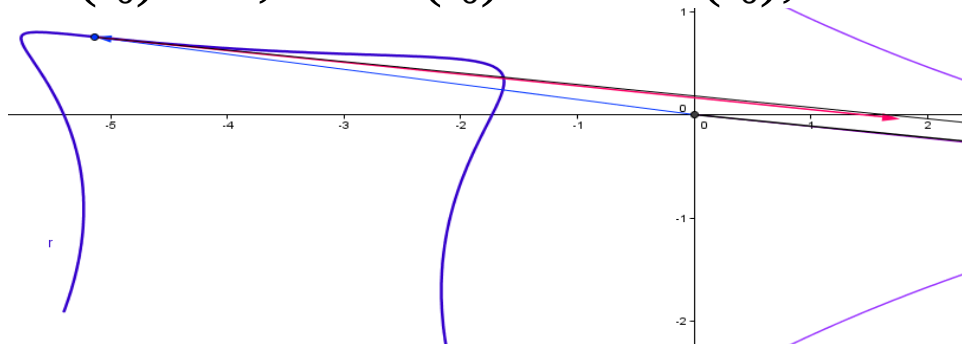
The following relations hold in a regular double point on curve

$$\mathbf{r}(t_0) = \mathbf{r}(t_1), \mathbf{r}'(t_0) \neq m \cdot \mathbf{r}'(t_1), m \in \mathbf{R}$$



In the inflex point on curve it holds

$$\mathbf{r}''(t_0) = \mathbf{0}, \text{ or } \mathbf{r}'(t_0) = m \cdot \mathbf{r}''(t_0), m \in \mathbf{R}$$



Length of line segment defined on $I = \langle a, b \rangle$ is value of integral

$$s = \int_a^b ds = \int_a^b \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} dt = \int_a^b \sqrt{d\mathbf{r} \cdot d\mathbf{r}}$$

Scalar function $s(t)$

$$s(t) = \int_a^t ds = \int_a^t \sqrt{d^2 \mathbf{r}}, t \in \langle a, b \rangle$$

Defines length of curve segment as function of variable t on interval I .

Properties of curves are described in regular points on curve segments, it means determining vector functions $\mathbf{r}(t)$ are at least once differentiable on domains (intervals I), so they have on I continuous derivatives up to order $n \geq 3$. There exist at least first three derivatives

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) \neq \mathbf{0}$$

$$\mathbf{r}''(t) = (x''(t), y''(t), z''(t)), \quad \mathbf{r}'''(t) = (x'''(t), y'''(t), z'''(t))$$

Elements of Frenet –Serret moving trihedron

Unit tangent vector of curve k in the point $P(t_0)$ is vector $\mathbf{t}(t_0) = \frac{\mathbf{r}'(t_0)}{|\mathbf{r}'(t_0)|}$

Vector $\mathbf{t}$ in arbitrary point $P(t)$, $t \in I$ can be represented also in the form

$$\mathbf{t} = c(t) \cdot \mathbf{r}', c(t) = \frac{1}{\sqrt{(\mathbf{r}' \cdot \mathbf{r}')}} ,$$

where coordinates of vector $\mathbf{t}$ are direction cosines of tangent line to the curve in this point

$$\mathbf{t} = (t_x, t_y, t_z), t_x = c(t)x'(t), t_y = c(t)y'(t), t_z = c(t)z'(t), t \in I$$

Equation of tangent to curve k in point $P(t_0)$, $t_0 \in I$
vector form

$$\mathbf{T}(u) = \mathbf{r}(t_0) + u \mathbf{t}(t_0), u \in \mathbf{R}$$

implicit form

$$\frac{X - x(t_0)}{x'(t_0)} = \frac{Y - y(t_0)}{y'(t_0)} = \frac{Z - z(t_0)}{z'(t_0)}$$

Unit binormal vector of curve k in the point $P(t_0)$ is vector

$$\mathbf{b}(t_0) = \frac{\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)}{|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)|}$$

Vector $\mathbf{b}$ in arbitrary point $P(t)$, $t \in I$ can be represented also in the form

$$\mathbf{b} = d(t) \cdot (\mathbf{r}' \times \mathbf{r}''), d(t) = \frac{1}{\sqrt{(\mathbf{r}' \times \mathbf{r}'') \cdot (\mathbf{r}' \times \mathbf{r}'')}} ,$$

where coordinates of vector $\mathbf{b}$ are direction cosines of binormal line to curve

$$\mathbf{b} = (b_x, b_y, b_z)$$

$$b_x = d(t) \begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix}, b_y = d(t) \begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix}, b_z = d(t) \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix}, t \in I$$

Equation of binormal to curve k in point $P(t_0)$, $t_0 \in I$

vector form $\mathbf{B}(u) = \mathbf{r}(t_0) + u \mathbf{b}(t_0)$, $u \in \mathbf{R}$

implicit form

$$\frac{X - x(t_0)}{\begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix}} = \frac{Y - y(t_0)}{\begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix}} = \frac{Z - z(t_0)}{\begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix}}$$

Unit (principle) normal vector to curve k in the point $P(t_0)$ is vector

$$\mathbf{n}(t_0) = \mathbf{b}(t_0) \times \mathbf{t}(t_0)$$

Vector $\mathbf{n}$ in arbitrary point $P(t)$, $t \in I$ can be also represented in the form

$$\mathbf{n} = \frac{(c'(t)\mathbf{r}' + c(t)\mathbf{r}'')}{a(t)}, c(t) = \frac{1}{\sqrt{(\mathbf{r}' \cdot \mathbf{r}')}},$$
$$a(t) = \sqrt{(c'(t)\mathbf{r}' + c(t)\mathbf{r}'') \cdot (c'(t)\mathbf{r}' + c(t)\mathbf{r}'')},$$

while coordinates of vector $\mathbf{n}$ are direction cosines of principle normal to curve in this point

$$n_x : n_y : n_z = [c'(t)x'(t)]' : [c'(t)y'(t)]' : [c'(t)z'(t)]', t \in I$$

Equation of normal to curve k in point $P(t_0)$, $t_0 \in I$

vector form $\mathbf{N}(u) = \mathbf{r}(t_0) + u \mathbf{n}(t_0)$, $u \in \mathbf{R}$

implicit form

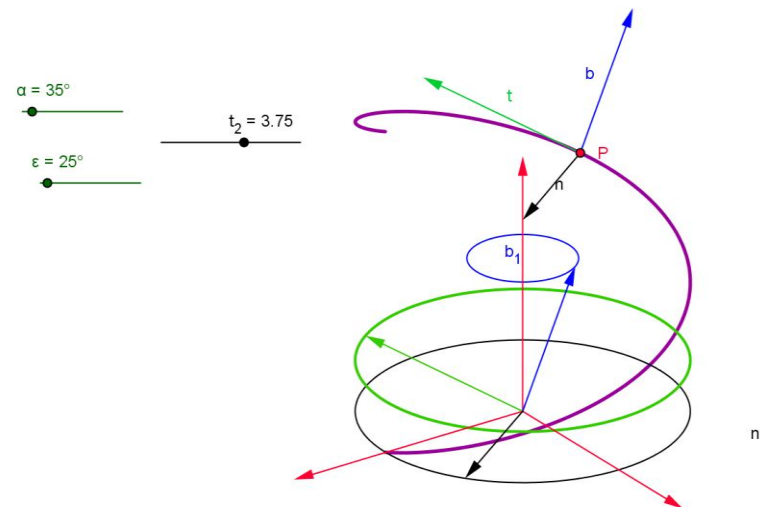
$$\frac{X - x(t_0)}{n_x} = \frac{Y - y(t_0)}{n_y} = \frac{Z - z(t_0)}{n_z}$$

Vectors $\mathbf{t}$, $\mathbf{b}$ and $\mathbf{n}$ in the regular point $P(t)$ on curve are 3 perpendicular unit vectors forming orthonormal reper of curve, called intrinsic orthogonal coordinate system moving together with the point on the curve and changing its position in the space accordingly, but keeping their mutual orthogonality. Lines - tangent, binormal and normal are 3 mutually perpendicular lines with one common point in regular point $P(t)$ forming moving trihedron of curve called also Frenet-Serret trihedron.

vertex – regular point $P(t)$ on curve

edges – halflines on tangent, principle normal and binormal

faces – in perpendicular planes $\omega = (P, \mathbf{t}, \mathbf{n})$, $\rho = (P, \mathbf{t}, \mathbf{b})$, $\nu = (P, \mathbf{n}, \mathbf{b})$



Osculating plane $\omega = (P, \mathbf{t}, \mathbf{n})$

Osculating plane in a regular point $P(t_0)$ on curve is perpendicular to the binormal in the given point, its direction vectors are unit tangent vector $\mathbf{t}$ and unit principle normal vector $\mathbf{n}$.

Vector equation

$$(\mathbf{R} - \mathbf{r}(t_0)) \cdot \mathbf{b}(t_0) = 0$$

$$[\mathbf{R} - \mathbf{r}, \mathbf{r}', \mathbf{r}''] = 0$$

$$\begin{vmatrix} X-x(t) & Y-y(t) & Z-z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \begin{vmatrix} X-x(t) & Y-y(t) & Z-z(t) \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix} = 0$$

Planar curve has the same osculating plane in all points, it is located in this unique osculating plane.

Space curve has a different osculating plane at each point $P(t_0)$ and it is located in this plane in a small neighbourhood of the respective point.

Rectification plane $\rho = (P, \mathbf{t}, \mathbf{b})$

Rectification plane in a regular point $P(t_0)$ on curve is perpendicular to the principle normal in the given point, its direction vectors are unit tangent vector $\mathbf{t}$ and unit binormal vector $\mathbf{b}$.

Vector equation

$$(\mathbf{R} - \mathbf{r}(t_0)) \cdot \mathbf{n}(t_0) = 0$$

$$[\mathbf{R} - \mathbf{r}, \mathbf{r}', \mathbf{r}' \times \mathbf{r}''] = 0$$

$$\begin{vmatrix} X-x(t) & Y-y(t) & Z-z(t) \\ x'(t) & y'(t) & z'(t) \\ \begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix} & \begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix} & \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix} \end{vmatrix} = 0$$

Rectification plane of a planar curve is perpendicular to the curve plane.

Length of a space curve k can be determined as length of a planar curve k^* that is development of curve k to its rectification plane.

Normal plane $\nu = (P, \mathbf{b}, \mathbf{n})$

Normal plane in a regular point $P(t_0)$ on curve is perpendicular to tangent in the given point, its direction vectors are unit vectors $\mathbf{b}$ and $\mathbf{n}$.

Vector equation $(\mathbf{R} - \mathbf{r}(t_0)) \cdot \mathbf{t}(t_0) =$

$$[\mathbf{R} - \mathbf{r}, \mathbf{r}' \times \mathbf{r}'', \mathbf{r}' \times (\mathbf{r}' \times \mathbf{r}'')] = 0$$

$$\begin{vmatrix} X-x(t) & Y-y(t) & Z-z(t) \\ \begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix} & \begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix} & \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix} \\ xr(t) & yr(t) & zr(t) \end{vmatrix} = 0$$

$$xr(t) = y'(t) \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix} - z'(t) \begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix}$$

$$yr(t) = x'(t) \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix} - z'(t) \begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix}$$

$$zr(t) = x'(t) \begin{vmatrix} x'(t) & z'(t) \\ x''(t) & z''(t) \end{vmatrix} - y'(t) \begin{vmatrix} y'(t) & z'(t) \\ y''(t) & z''(t) \end{vmatrix}$$

Deviation of curve from its tangent in a regular point $P(t_0)$ is represented by the first curvature - flexion.

Function of the first curvature

$${}^1k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, t \in I$$

is scalar non-negative function.

Osculating circle $h(S, \rho)$ to curve is located in the osculating plane in the respective regular point $P(t_0)$ and it has a common tangent with the curve.

Centre of osculating circle is a point on principle normal in the point $P(t_0)$ in direction of unit normal vector.

Radius of osculating circle = radius of the first curvature in the given point, and it equals to inverted value of function ${}^1k(t)$ for respective t_0 .

$$\rho = \frac{1}{{}^1k(t_0)}$$

Regular point on curve, in which ${}^1k(t) = 0$, is inflexion point, as $\mathbf{r}''(t) = \mathbf{0}$.
 If ${}^1k(t) = 0$ in all points on curve, curver is a line (line segment).

$$\mathbf{r}(t) = (at + b, ct + d, et + f), t \in \mathbf{R}$$

$$\mathbf{r}'(t) = (a, c, e), \mathbf{r}''(t) = (0, 0, 0)$$

$$|\mathbf{r}'(t)| = \sqrt{a^2 + c^2 + e^2}, \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{0}, |\mathbf{r}'(t) \times \mathbf{r}'(t)| = 0$$

$${}^1k(t) = \frac{0}{\sqrt{a^2 + c^2 + e^2}} = 0, \forall t \in I$$

Curve with constant non-zero first curvature is e.g. circle.

$$\mathbf{r}(t) = (r \cos t, r \sin t, 0), r \in \mathbf{R}, t \in \langle 0, 2\pi \rangle$$

$$\mathbf{r}'(t) = (-r \sin t, r \cos t, 0), \mathbf{r}''(t) = (-r \cos t, -r \sin t, 0),$$

$$\mathbf{r}'''(t) = (r \sin t, -r \cos t, 0), \mathbf{r}^{(4)}(t) = (r \cos t, r \sin t, 0)$$

$$|\mathbf{r}'(t)| = r, \mathbf{r}'(t) \times \mathbf{r}''(t) = (0, 0, r^2), |\mathbf{r}'(t) \times \mathbf{r}''(t)| = r^2$$

$${}^1k(t) = \frac{r^2}{r^3} = \frac{1}{r}, \forall t \in \langle 0, 2\pi \rangle$$

Circle is self-osculating to itself in all points.

Deviation of curve from its osculating plane in a regular point $P(t_0)$ is represented by the second curvature - torsion.

Function of the second curvature

$$^2k(t) = \frac{[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)]}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}, t \in I$$

is a scalar function.

Sign of value of function $^2k(t)$ in the point t_0 determines orientation of deviation of curve from the osculating plane in the point $P(t_0)$.

$^2k(t_0) > 0$, curve is deviating in the direction of unit binormal vector,
over the osculating plane in given point

$^2k(t_0) < 0$, curve is deviating in the opposite direction to the direction of unit
binormal vector, below osculating plane in given point

$^2k(t_0) = 0$, point is called planar point, curve is not deviating from osculating
plane in given point, it is located in this plane in the point neighbourhood

If $^2k(t) = 0$ in all points on curve, curve is in one plane, i.e. it is planar.

For circle holds

$${}^2k(t) = \frac{\begin{vmatrix} -r \sin t & r \cos t & 0 \\ -r \cos t & -r \sin t & 0 \\ r \sin t & -r \cos t & 0 \end{vmatrix}}{r^4} = 0$$

Curve of constant non-zero second curvature is e.g. helix.

$$\mathbf{r}(t) = (r \cos t, r \sin t, vt), r, v \in \mathbf{R}, t \in \langle 0, 2\pi \rangle$$

$$\mathbf{r}'(t) = (-r \sin t, r \cos t, v), \mathbf{r}''(t) = (-r \cos t, -r \sin t, 0), \mathbf{r}'''(t) = (r \sin t, -r \cos t, 0)$$

$$|\mathbf{r}'(t)| = \sqrt{r^2 + v^2}, \mathbf{r}'(t) \times \mathbf{r}''(t) = (rv \sin t, rv \cos t, r^2), |\mathbf{r}'(t) \times \mathbf{r}''(t)| = r\sqrt{r^2 + v^2}$$

$$[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)] = vr^2$$

$${}^1k(t) = \frac{r\sqrt{r^2 + v^2}}{(r^2 + v^2)\sqrt{r^2 + v^2}} = \frac{r}{r^2 + v^2}, \forall t \in \langle 0, 2\pi \rangle$$

$${}^2k(t) = \frac{vr^2}{r^2(r^2 + v^2)} = \frac{v}{r^2 + v^2} > 0, \forall t \in \langle 0, 2\pi \rangle$$

Two curves $\mathbf{r}(t), \mathbf{p}(t), t \in I$ are called **equidistant** (parallel) **curves**, if there exists number $c \in \mathbf{R}$ such, that $\forall t \in I$ it holds

$$\mathbf{p}(t) = \mathbf{r}(t) + c.\mathbf{n}(t)$$

Any 2 concentric circles are equidistant, or 2 ellipses with common centre, while for their semi-axes holds relation $a_2 = c.a_1, b_2 = c.b_1$, for small c .

Curves of constant slope

Let ω be a fixed angle determined by direction - gradient $(\cos \omega, \sin \omega)$. Space curve, whose functions of the first and the second curvature satisfy equation

$${}^2k \sin \omega - {}^1k \cos \omega = 0$$

are called curves with constant slope, while slope is the tangents of angle ω , and it holds

$$\tan \omega = \frac{{}^1k}{{}^2k} = \text{const}$$

Cylindrical helix is a curve of constant slope, $\tan \omega = r/v$.

Frenet-Serret formulas

represent relations between unit tangent, normal and binormal vectors and their first derivatives in a curve regular point

$$\mathbf{t}' = {}^1k|\mathbf{r}'|\mathbf{n}$$

$$\mathbf{n}' = -{}^1k|\mathbf{r}'|\mathbf{t} + {}^2k|\mathbf{r}'|\mathbf{b}$$

$$\mathbf{b}' = -{}^2k|\mathbf{r}'|\mathbf{n}$$

$${}^1k = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, {}^2k = \frac{[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']}{|\mathbf{r}' \times \mathbf{r}''|^2}$$