Operations with vector functions

Let vector functions $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and scalar function $\varphi(t)$ be defined on the set M.

For any real number t_0 the function values $\mathbf{r}(t_0)$, $\mathbf{r}_1(t_0)$, $\mathbf{r}_2(t_0)$ are vectors and function value $\varphi(t_0)$ is a scalar (number).

The following values can be attached to any real number $t_0 \in M$

1. vector $\mathbf{r}_1(t_0) + \mathbf{r}_2(t_0)$
2. vector $\varphi(t_0)\mathbf{r}(t_0)$
3. scalar $\mathbf{r}_1(t_0).\mathbf{r}_2(t_0)$
4. vector $\mathbf{r}_1(t_0) \times \mathbf{r}_2(t_0)$ for three dimensional functions

The following functions are defined on M in this way

1. sum of vector functions - vector function $\mathbf{r}_1(t) + \mathbf{r}_2(t)$
2. product of scalar and vector function – vector function $\varphi(t)\mathbf{r}(t)$
3. scalar product of vector functions - scalar function $\mathbf{r}_1(t).\mathbf{r}_2(t)$
4. vector product of vector functions – vector function $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$

Let

$\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$

$\mathbf{r}_1(t) = (g_1(t), g_2(t), \ldots, g_n(t))$

$\mathbf{r}_2(t) = (h_1(t), h_2(t), \ldots, h_n(t))$

Then

1. $\mathbf{r}_1(t) + \mathbf{r}_2(t) = (g_1(t) + h_1(t), g_2(t) + h_2(t), \ldots, g_n(t) + h_n(t))$
2. $\varphi(t)\mathbf{r}(t) = (\varphi(t)f_1(t), \varphi(t)f_2(t), \ldots, \varphi(t)f_n(t))$
3. $\mathbf{r}_1(t).\mathbf{r}_2(t) = g_1(t)h_1(t) + g_2(t)h_2(t) + \ldots + g_n(t)h_n(t)$ = scalar function
4. for $\mathbf{r}_1(t) = (x_1(t), y_1(t), z_1(t))$, $\mathbf{r}_2(t) = (x_2(t), y_2(t), z_2(t))$

$$\mathbf{r}_1(t) \times \mathbf{r}_2(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \end{vmatrix} =$$

$$= (y_1(t)z_2(t) - y_2(t)z_1(t), x_2(t)z_1(t) - x_1(t)z_2(t), x_1(t)y_2(t) - x_2(t)y_1(t))$$

Let vector functions $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and scalar function $\varphi(t)$ have in the point $t_0 \in M$ their limits - vectors $\mathbf{a}$, $\mathbf{a}_1$, $\mathbf{a}_2$ and number k, then also functions $\mathbf{r}_1(t) + \mathbf{r}_2(t)$, $\varphi(t)\mathbf{r}(t)$, $\mathbf{r}_1(t).\mathbf{r}_2(t)$ and $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$ have at the point $t_0 \in M$ limits and it holds

1. $$\lim_{t\to t_0}[\mathbf{r}_1(t)+\mathbf{r}_2(t)] = \lim_{t\to t_0}\mathbf{r}_1(t) + \lim_{t\to t_0}\mathbf{r}_2(t) = \mathbf{a}_1 + \mathbf{a}_2$$

2. $$\lim_{t\to t_0}[\varphi(t)\mathbf{r}(t)] = \left[\lim_{t\to t_0}\varphi(t)\right].\left[\lim_{t\to t_0}\mathbf{r}(t)\right] = k\mathbf{a}$$

3. $$\lim_{t\to t_0}[\mathbf{r}_1(t).\mathbf{r}_2(t)] = \left[\lim_{t\to t_0}\mathbf{r}_1(t)\right].\left[\lim_{t\to t_0}\mathbf{r}_2(t)\right] = \mathbf{a}_1.\mathbf{a}_2$$

4. $$\lim_{t\to t_0}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \left[\lim_{t\to t_0}\mathbf{r}_1(t)\right]\times\left[\lim_{t\to t_0}\mathbf{r}_2(t)\right] = \mathbf{a}_1\times\mathbf{a}_2$$

Let vector functions $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and scalar function $\varphi(t)$ be continuous in the point $t_0 \in M$, then also functions $\mathbf{r}_1(t) + \mathbf{r}_2(t)$, $\varphi(t)\mathbf{r}(t)$, $\mathbf{r}_1(t).\mathbf{r}_2(t)$ and $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$ are continuous at the point $t_0 \in M$.

Let vector functions $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and scalar function $\varphi(t)$ have derivatives on the set M , then also functions $\mathbf{r}_1(t) + \mathbf{r}_2(t)$, $\varphi(t)\mathbf{r}(t)$, $\mathbf{r}_1(t).\boldsymbol{r}_2(t)$ and $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$ have derivatives on the set M and it holds

1. $[\mathbf{r}_1(t) + \mathbf{r}_2(t)]' = \mathbf{r}_1{}'(t) + \mathbf{r}_2{}'(t) =$
$$= (g_1{}'(t) + h_1{}'(t), g_2{}'(t) + h_2{}'(t), \ldots, g_n{}'(t) + h_n{}'(t))$$
2. $[\varphi(t)\mathbf{r}(t)]' = \varphi'(t)\mathbf{r}(t) + \varphi(t)\mathbf{r}'(t) =$
$$= (\varphi'(t)g_1(t)+\varphi(t)g_1{}'(t), \varphi'(t)g_2(t)+\varphi(t)g_2{}'(t),...,\varphi'(t)g_n(t)+\varphi(t)g_n{}'(t))$$
3. $[\mathbf{r}_1(t).\mathbf{r}_2(t)]' = \mathbf{r}_1{}'(t).\mathbf{r}_2(t) + \mathbf{r}_1(t).\mathbf{r}_2{}'(t) =$
$$= g_1{}'.h_1 + g_1.h_1{}' + g_2{}'.h_2 + g_2.h_2{}' + \ldots + g_n{}'.h_n + g_n.h_n{}'$$
4. $[\mathbf{r}_1(t) \times \mathbf{r}_2(t)]' = \mathbf{r}_1{}'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times .\mathbf{r}_2{}'(t)$

Rules for finding derivatives of vector operations

Let vector functions $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and scalar function $\varphi(t)$ have continuous derivatives on the set M, then also functions $\mathbf{r}_1(t) + \mathbf{r}_2(t)$, $\varphi(t)\mathbf{r}(t)$, $\mathbf{r}_1(t).\mathbf{r}_2(t)$ and $\mathbf{r}_1(t) \times \mathbf{r}_2(t)$ have continuous derivatives on the set M and it holds

1. $[\mathbf{r}_1(t) + \mathbf{r}_2(t)]'' = \mathbf{r}_1''(t) + \mathbf{r}_2''(t)$

2. $[\varphi(t)\mathbf{r}(t)]'' = [\varphi'(t)\mathbf{r}(t) + \varphi(t)\mathbf{r}'(t)]' =$

$$= \varphi''(t)\mathbf{r}(t) + 2\varphi'(t)\mathbf{r}'(t) + \varphi(t)\mathbf{r}''(t)$$

3. $[\mathbf{r}_1(t).\mathbf{r}_2(t)]'' = [\mathbf{r}_1'(t).\mathbf{r}_2(t) + \mathbf{r}_1(t).\mathbf{r}_2'(t)]' =$

$$= \mathbf{r}_1''(t).\mathbf{r}_2(t) + 2\mathbf{r}_1'(t).\mathbf{r}_2'(t) + \mathbf{r}_1(t).\mathbf{r}_2''(t)$$

4. $[\mathbf{r}_1(t) \times \mathbf{r}_2(t)]'' = [\mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times .\mathbf{r}_2'(t)]' =$

$$= \mathbf{r}_1''(t) \times \mathbf{r}_2(t) + 2\mathbf{r}_1'(t) \times \mathbf{r}_2'(t) + \mathbf{r}_1(t) \times .\mathbf{r}_2''(t)$$

All presented propertiers valid for operations on vector functions of one real variable are valid also for vector functions of more variables.

Vector function $\mathbf{f}(X) = (f_1(X), f_2(X), \ldots, f_n(X))$, $X = (t_1, t_2, \ldots, t_n)$ has a limit $\mathbf{a}$ at the point X_0 iff all scalar coordinate functions $f_1(X), f_2(X), \ldots, f_n(X)$ have limits at this point and it holds

$$\lim_{X \to X_0} f_1(X) = a_1, \lim_{X \to X_0} f_2(X) = a_2, \ldots, \lim_{X \to X_0} f_n(X) = a_n$$

$$\lim_{X \to X_0} \mathbf{f}(X) = \lim_{X \to X_0} [(f_1(X), f_2(X), \ldots, f_n(X))] =$$
$$= (\lim_{x \to X_0} f_1(X), \lim_{X \to X_0} f_2(X), \ldots, \lim_{X \to X_0} f_n(X)) = (a_1, a_2, \ldots, a_n) = \boldsymbol{a}$$

Limit of vector function $\mathbf{f}(X)$ at the point X_0 is vector $\mathbf{a}$, whose components are limits of scalar coordinate functions $f_i(X)$ at the point X_0.

Vector function $\mathbf{f}(X) = (f_1(X), f_2(X), \ldots, f_n(X)), X = (t_1, t_2, \ldots, t_n)$

has partial derivative with respect to variable t_j at the point A iff all scalar coordinate functions $f_1(X), f_2(X), \ldots, f_n(X)$ have partial derivatives at this point and it holds

$$\mathbf{f}'_{t_j}(A) = \frac{\partial \mathbf{f}(A)}{\partial t_j} = \left(\frac{\partial f_1(A)}{\partial t_j}, \frac{\partial f_2(A)}{\partial t_j}, \ldots, \frac{\partial f_n(A)}{\partial t_j} \right)$$

Analogously, partial derivatives of vector functions of more variables can be defined as functions on domain M and range in the vector space $\boldsymbol{V}^n(\boldsymbol{R})$.

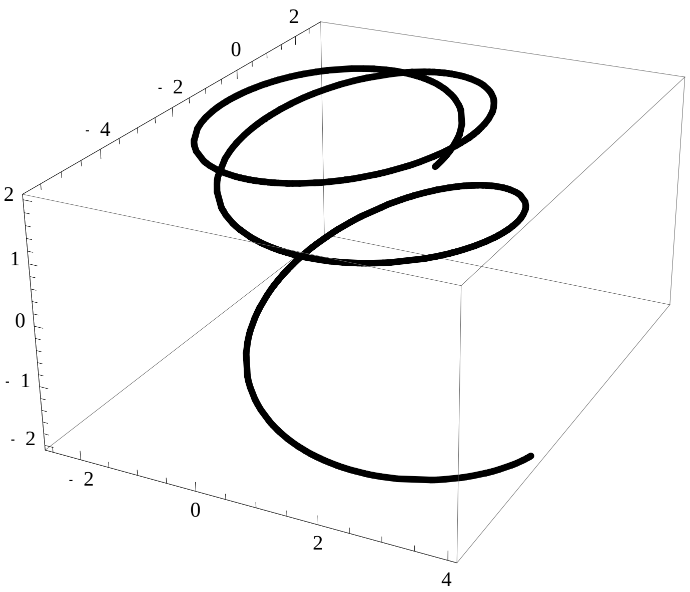
$$\mathbf{f}'_{t_j}(X) = \frac{\partial \mathbf{f}(X)}{\partial t_j} = \left(\frac{\partial f_1(X)}{\partial t_j}, \frac{\partial f_2(X)}{\partial t_j}, \ldots, \frac{\partial f_n(X)}{\partial t_j} \right)$$

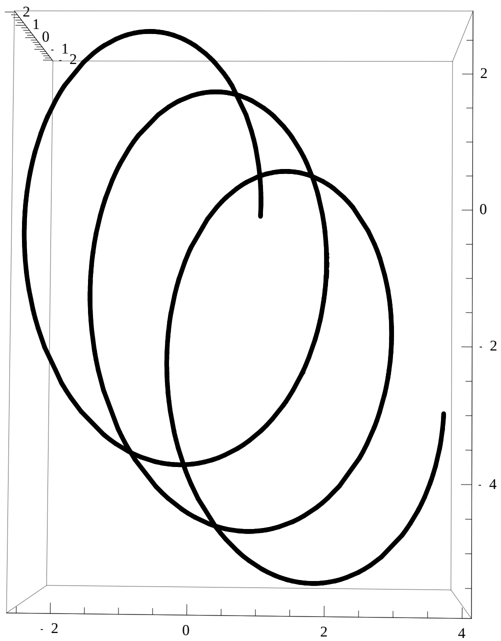
Three-dimensional vector function of one variable defined on interval in real numbers, for all $t \in I \subset \boldsymbol{R}$

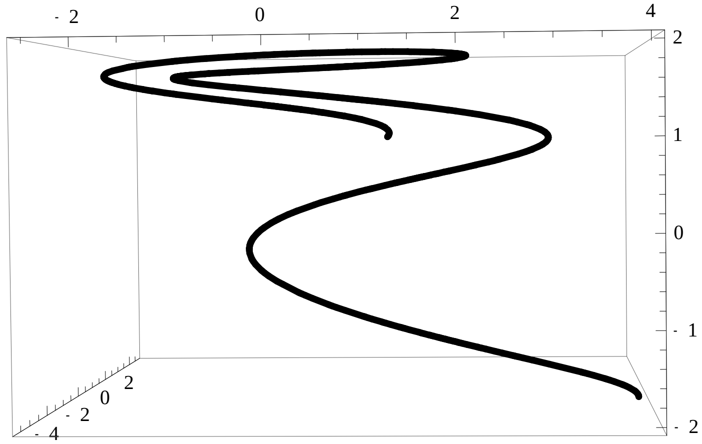
$$\mathbf{f}(t) = x(t).\mathbf{i} + y(t).\mathbf{j} + z(t).\boldsymbol{k} = (x(t), y(t), z(t))$$

hodograph is a curve in space $\boldsymbol{E}^3$

$$\mathbf{f}\,'(t) = x'(t).\mathbf{i} + y'(t).\mathbf{j} + z'(t).\mathbf{k} = (x'(t), y'(t), z'(t))$$

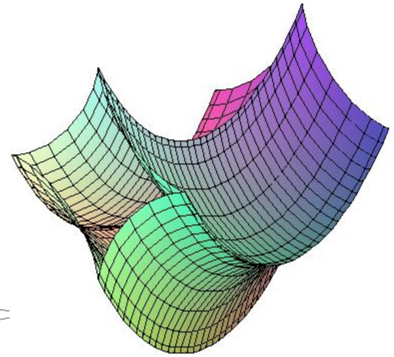






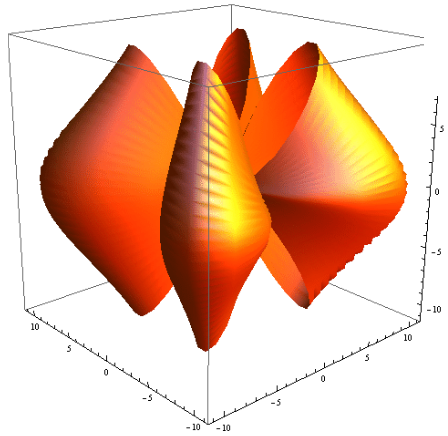
Three-dimensional vector function with two variables

defined on planar region, for all $(u, v) \in \Omega \subset \boldsymbol{R}^2$



$$\mathbf{f}(u, v) = x(u, v).\mathbf{i} + y(u, v).\mathbf{j} + z(u, v).\mathbf{k} =$$

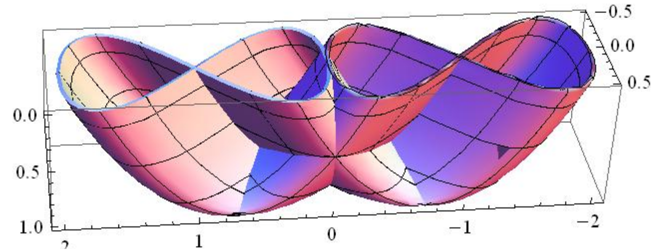
$$= (x(u, v), y(u, v), z(u, v))$$



Hodograph is a surface in the space $\boldsymbol{E}^3$

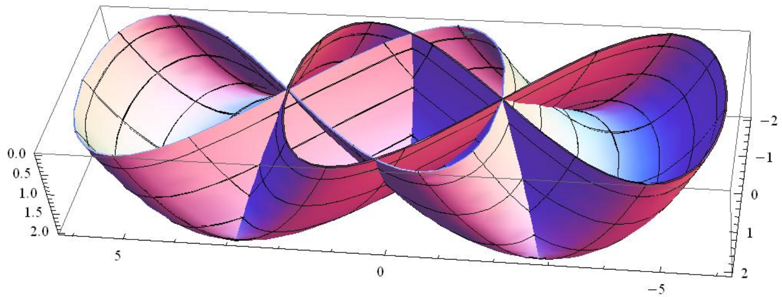
$$\mathbf{f}_u(u, v) = x_u(u, v).\mathbf{i} + y_u(u, v).\mathbf{j} + z_u(u, v).\mathbf{k} =$$

$$= (x_u(u, v), y_u(u, v), z_u(u, v))$$



$$\mathbf{f}_v(u, v) = x_v(u, v).\mathbf{i} + y_v(u, v).\mathbf{j} + z_v(u, v).\mathbf{k} =$$

$$= (x_v(u, v), y_v(u, v), z_v(u, v))$$



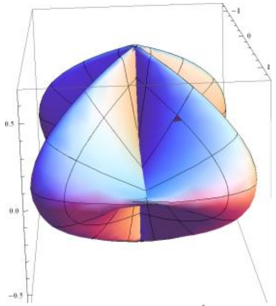
Three-dimensional vector function with three variables

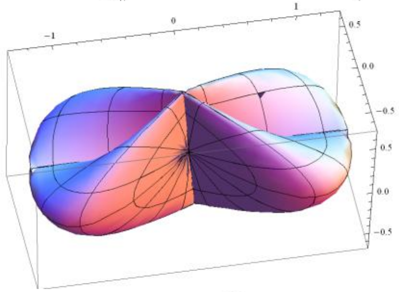
Defined on space region, for all $(u, v, w) \in \Omega \subset \boldsymbol{R}^3$

$$\mathbf{f}(u, v, w) = x(u, v, w).\mathbf{i} + y(u, v, w).\mathbf{j} + z(u, v, w).\mathbf{k} =$$

$$= (x(u, v, w), y(u, v, w), z(u, v, w))$$

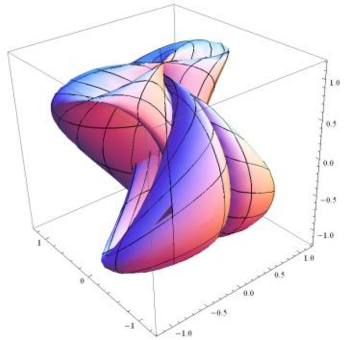
hodograph is a solid in the space $\boldsymbol{E}^3$





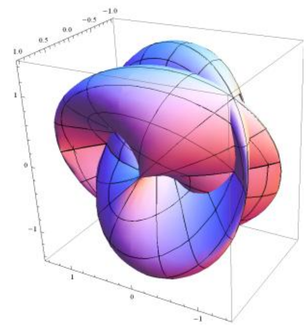
$$\mathbf{f}_u(u, v, w) = x_u(u, v, w).\mathbf{i} + y_u(u, v, w).\mathbf{j} + z_u(u, v, w).\boldsymbol{k} =$$

$$= (x_u(u, v, w), y_u(u, v, w), z_u(u, v, w))$$



$$\mathbf{f}_v(u, v, w) = x_v(u, v, w).\mathbf{i} + y_v(u, v, w).\mathbf{j} + z_v(u, v, w).\mathbf{k} =$$

$$= (x_v(u, v, w), y_v(u, v, w), z_v(u, v, w))$$



$$\mathbf{f}_w(u, v, w) = x_w(u, v, w).\mathbf{i} + y_w(u, v, w).\mathbf{j} + z_w(u, v, w).\boldsymbol{k} =$$

$$= (x_w(u, v, w), y_w(u, v, w), z_w(u, v, w))$$