Integration of vector functions

Taylor theorem

Let vector function $\mathbf{r}(t)$ has continuous derivatives up to oder $n + 1$ (it is differentiable) on a closed interval $J = \langle t_0 , t_0 + \Delta t\rangle$.
Then there exists vector function $\boldsymbol{\alpha}$ such that

$$\lim_{\Delta t \to 0} \boldsymbol{\alpha}(\Delta t) = \boldsymbol{0}$$

and it holds

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \frac{\mathbf{r}'(t_0)}{1!}\Delta t + \frac{\mathbf{r}''(t_0)}{2!}\Delta t^2 + \ldots + \frac{\mathbf{r}^{(n)}(t_0)}{n!}\Delta t^n + \boldsymbol{R}_{n+1}$$

$$\boldsymbol{R}_{n+1} = \left[\frac{\mathbf{r}^{(n+1)}(t_0)}{(n+1)!} + \boldsymbol{\alpha}(\Delta t) \right] \Delta t^{n+1}$$

This representation of function is called Taylor approximation of function of order n, where $\boldsymbol{R}_{n+1}$ is called error.

Integral of vector function

Vector function $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$

is integrable on interval $\langle \alpha, \beta \rangle$ iff,

all its components, scalar coordinate functions $f_1(t), f_2(t), \ldots, f_n(t)$ are

integrable on interval $\langle \alpha, \beta \rangle$, and it holds

$$\int_{\alpha}^{\beta} \mathbf{r}(t)\,dt = \left(\int_{\alpha}^{\beta} f_1(t)\,dt, \int_{\alpha}^{\beta} f_2(t)\,dt, \ldots, \int_{\alpha}^{\beta} f_n(t)\,dt \right)$$

Definite integral of vector function is vector from the space $V^n(\boldsymbol{R})$.

Vector function $\mathbf{R}(t) = (F_1(t), F_2(t), \dots, F_n(t))$

defined on an open interval $J = \langle \alpha, \beta \rangle$ is called antiderivative, or a primitive function of the vector function $\mathbf{r}(t)$ defined on the same interval J, if for any $t \in J$ holds $\mathbf{R}'(t) = \mathbf{r}(t)$.

Let functions $F_1(t), F_2(t), \dots, F_n(t)$ are antiderivatives of respective scalar coordinate functions $f_1(t), f_2(t), \dots, f_n(t)$.

Vector function $\mathbf{R}(t)$ is said to be indefinite integral of vector function $\mathbf{r}(t)$

$$\int \mathbf{r}(t)dt = \left(\int f_1(t)dt, \int f_2(t)dt, \dots, \int f_n(t)dt\right)$$

Let vector function $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$

be integrable on interval $J = \langle \alpha, \beta \rangle$

and let vector function $\mathbf{R}(t) = (F_1(t), F_2(t), \ldots, F_n(t))$

continuous on interval J be antiderivative of vector function $\mathbf{r}(t)$ on J.

Then it holds

$$\int_{\alpha}^{\beta} \mathbf{r}(t)dt = \mathbf{R}(\beta) - \mathbf{R}(\alpha) =$$

$$= (F_1(\alpha) - F_1(\beta), F_2(\alpha) - F_2(\beta), \ldots, F_n(\alpha) - F_n(\beta))$$

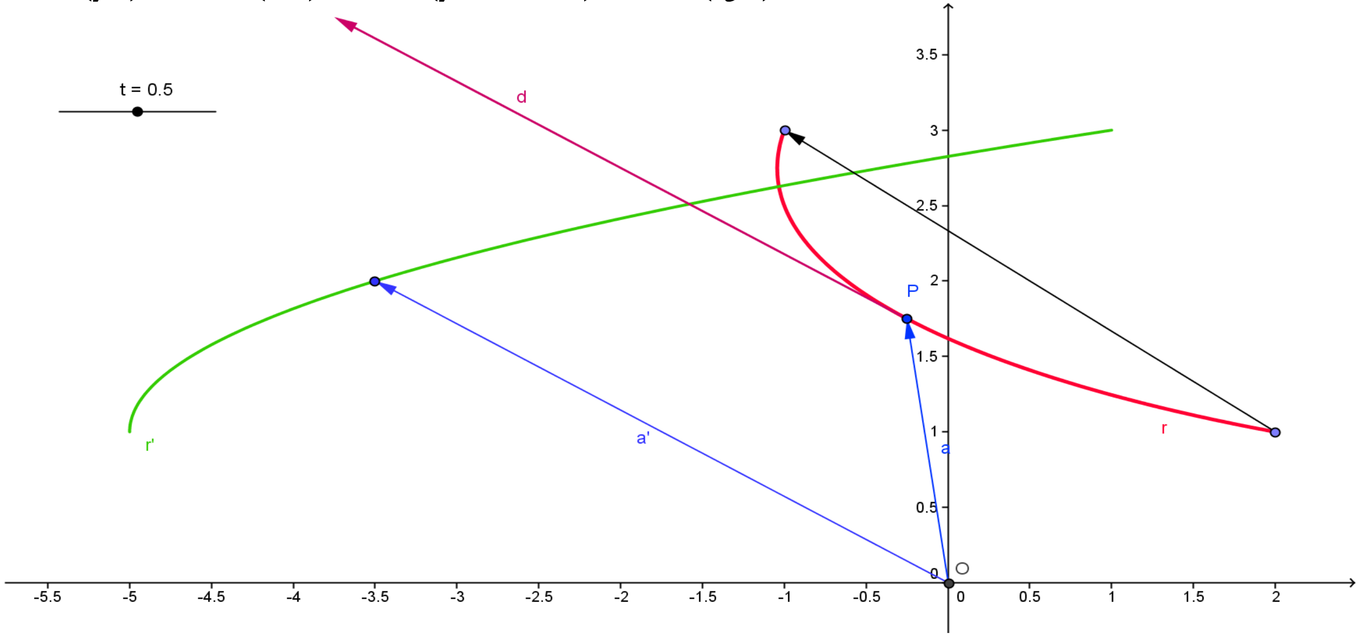
Newton-Leibnitz formula for vector functions

Let vector function **r** has the following properties:

1. it is continuous on interval $\langle \alpha, \beta \rangle$
2. it is differentiable at all points from interval (α, β).

Then there exists at least one point ξ in the interval $\langle \alpha, \beta \rangle$ such that

$$\left| \mathbf{r}(\beta) - \mathbf{r}(\alpha) \right| \le (\beta - \alpha) \left| \mathbf{r}'(\xi) \right|$$



Mean value theorem for vector function on closed interval