# Properties of vector functions

## Limit of a vector function

Let **r** $= (f_1, f_2, \ldots, f_n)$ be a vector function of one variable (scalar) $t$ and domain of definition $M \subset \boldsymbol{R}$ and let $t_0 \in M$ be such point (number), that one of its neighbourhoods (possibly without $t_0$) is a part of $M$, therefore function need not be defined at the point $t_0$.
Vector function $\boldsymbol{r}(t)$ is said to have limit $\boldsymbol{a}$ at the point $t_0$ denoted

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{a}$$

if for any sequence of real numbers $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} t_n = t_0, t_n \neq t_0, t_n \in M$$

the respective sequence of function values converges to vector $\boldsymbol{a}$

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{a} \Leftrightarrow \left[ \lim_{n \to \infty} t_n = t_0, t_n \neq t_0, t_n \in M \Rightarrow \lim_{n \to \infty} \mathbf{r}(t_n) = \mathbf{a} \right]$$

One-sided limits of vector function $\mathbf{r}(t)$ can be defined similarly, while

condition $t_n \neq t_0$ is substituted by condition:

$t_n < t_0$ for limit of vector function $\mathbf{r}(t)$ at the point $t_0$ from the left

$$\lim_{t \to t_0^-} \mathbf{r}(t) = \mathbf{a}$$

$t_n > t_0$ for limit of vector function $\mathbf{r}(t)$ at the point $t_0$ from the right

$$\lim_{t \to t_0^+} \mathbf{r}(t) = \mathbf{b}$$

Vector function $\mathbf{r}(t)$ has a limit at the point $t_0$ if and only if there exists limit from the left and limit from the right at this point and these two limits are equal

$$\lim_{t \to t_0} \mathbf{r}(t) = \lim_{t \to t_0^-} \mathbf{r}(t) = \mathbf{a} = \lim_{t \to t_0^+} \mathbf{r}(t) = \mathbf{b}$$

Vector function $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ has limit $\mathbf{a}$ at the point $t_0$ if and only if all scalar functions $f_1(t), f_2(t), \ldots, f_n(t)$ have limits at this point and holds

$$\lim_{t\to t_0} f_1(t) = a_1, \lim_{t\to t_0} f_2(t) = a_2, \ldots, \lim_{t\to t_0} f_n(t) = a_n$$

$$\lim_{t\to t_0} \mathbf{r}(t) = \lim_{t\to t_0}[(f_1(t), f_2(t), \ldots, f_n(t))] =$$
$$= (\lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \ldots, \lim_{t\to t_0} f_n(t)) = (a_1, a_2, \ldots, a_n) = \mathbf{a}$$

Limit of a vector function $\mathbf{r}(t)$ at the point $t_0$ is vector $\mathbf{a}$, whose components are equal to limits of scalar coordinate functions $f_i(t)$ at the point $t_0$.

Vector function $\mathbf{r}(t)$ is continuous at the point $t_0$ if for its limit holds

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

Limit of function equals to function value at this point, therefore function is defined at this point.

Analogously as one-sided limits continuity of a vector function at the point $t_0$ from the left can be defined

$$\lim_{t \to t_0^-} \mathbf{r}(t) = \mathbf{r}(t_0)$$

and continuity of a vector function at the point $t_0$ from the right

$$\lim_{t \to t_0^+} \mathbf{r}(t) = \mathbf{r}(t_0)$$

Vector function $\mathbf{r}(t)$ is continuous at the point $t_0$ if and only if it is continuous from the right and from the left at this point.

## Derivative of a vector function

Let $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ be a vector function of variable (scalar) $t$ with domain of definition $M \subset \boldsymbol{R}$ and let $t_0 \in M$ be such point (number) from $M$, that some of its neighbourhoods is a part of $M$, therefore function **is defined** at the point $t_0$.
If there exists limit

$$\lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}$$

it is said to be derivative of vector function at the point $t_0$ denoted

$$\mathbf{r}'(t_0) = \left[\frac{d\mathbf{r}}{dt}\right]_{t=t_0} = \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}$$

Vector function $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ has derivative $\mathbf{r}'(t_0)$ at the point $t_0$ if and only if all scalar functions $f_1(t)$, $f_2(t)$, …, $f_n(t)$ have derivatives $f_1'(t_0)$, $f_2'(t_0)$, …, $f_n'(t_0)$ at this point and it holds

$$\mathbf{r}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0} =$$

$$= \left( \lim_{t \to t_0} \frac{f_1(t) - f_1(t_0)}{t - t_0}, \lim_{t \to t_0} \frac{f_2(t) - f_2(t_0)}{t - t_0}, \ldots, \lim_{t \to t_0} \frac{f_n(t) - f_n(t_0)}{t - t_0} \right) =$$

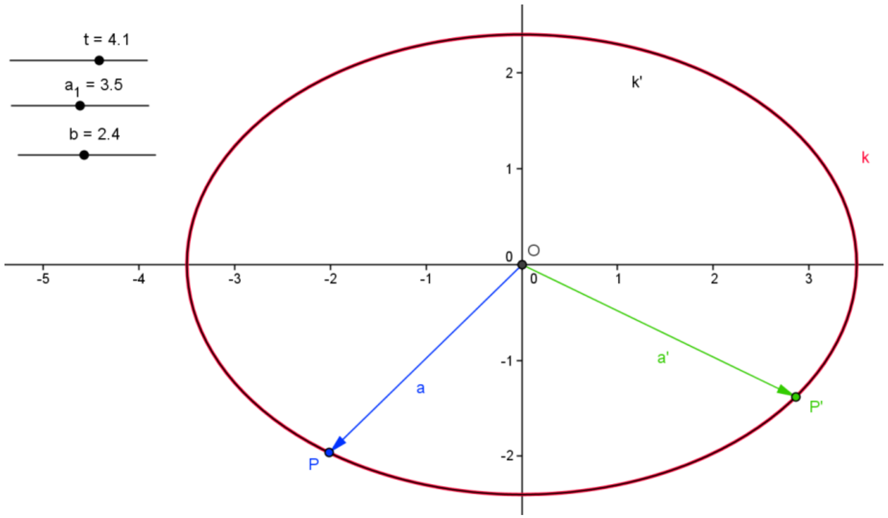
$$= (f_1'(t_0), f_2'(t_0), \ldots, f_n'(t_0))$$

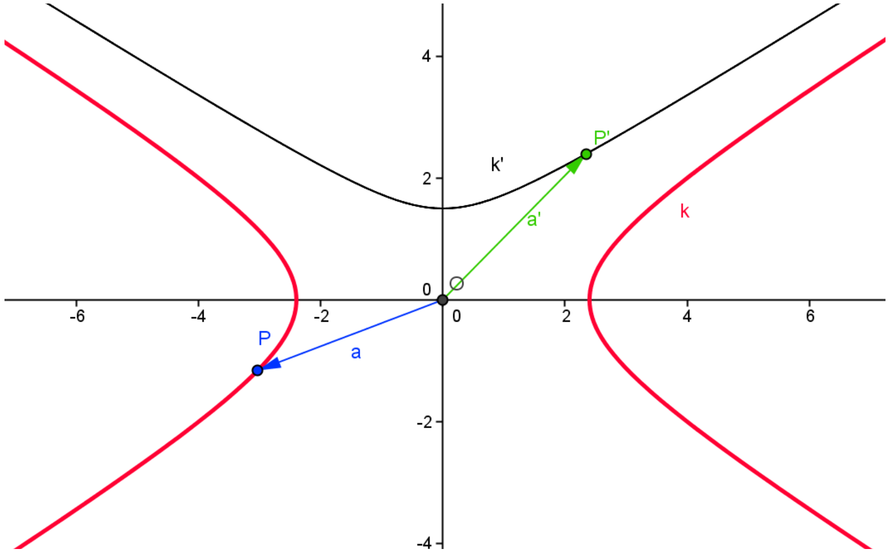
Derivative $\mathbf{r}'(t_0)$ of vector function $\mathbf{r}(t)$ at the point $t_0$ is vector, whose components ar equal to derivatives $f_i'(t_0)$ of respective scalar coordinate functions $f_i(t)$ at the point $t_0$.

If vector function $\mathbf{r}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ has derivative at all points of its domain of definition $M$, then it has derivative on $M$, which is a vector function defined on $M$ and it holds

$$\mathbf{r}'(t) = (f_1'(t), f_2'(t), \ldots, f_n'(t)), \; t \in M$$

Graph of derivative $\mathbf{r}'(t)$ of vector function $\mathbf{r}(t)$ is a curve $k'$, called also derivative of a curve $k$, which is graph of vector function $\mathbf{r}(t)$.



Derivatives of higher orders are defined analogously

$$\mathbf{r}''(t_0)=\left[\frac{d^2\mathbf{r}}{dt^2}\right]_{t=t_0}=\lim_{t\to t_0}\frac{\mathbf{r}'(t)-\mathbf{r}'(t_0)}{t-t_0}=$$

$$=(f_1''(t_0), f_2''(t_0),\dots,f_n''(t_0))$$

$$\mathbf{r}'''(t_0)=\left[\frac{d^3\mathbf{r}}{dt^3}\right]_{t=t_0}=\lim_{t\to t_0}\frac{\mathbf{r}''(t)-\mathbf{r}''(t_0)}{t-t_0}=$$ , and so on, for higher orders

$$=(f_1'''(t_0), f_2'''(t_0),\dots,f_n'''(t_0))$$

**Geometric interpretation of derivative of vector function at point**

$\mathbf{r}'(t_0)$ is direction vector of tangent line to graph of vector function $\mathbf{r}(t)$

For two-dimensionl vector function defined on $I \subset \boldsymbol{R}$

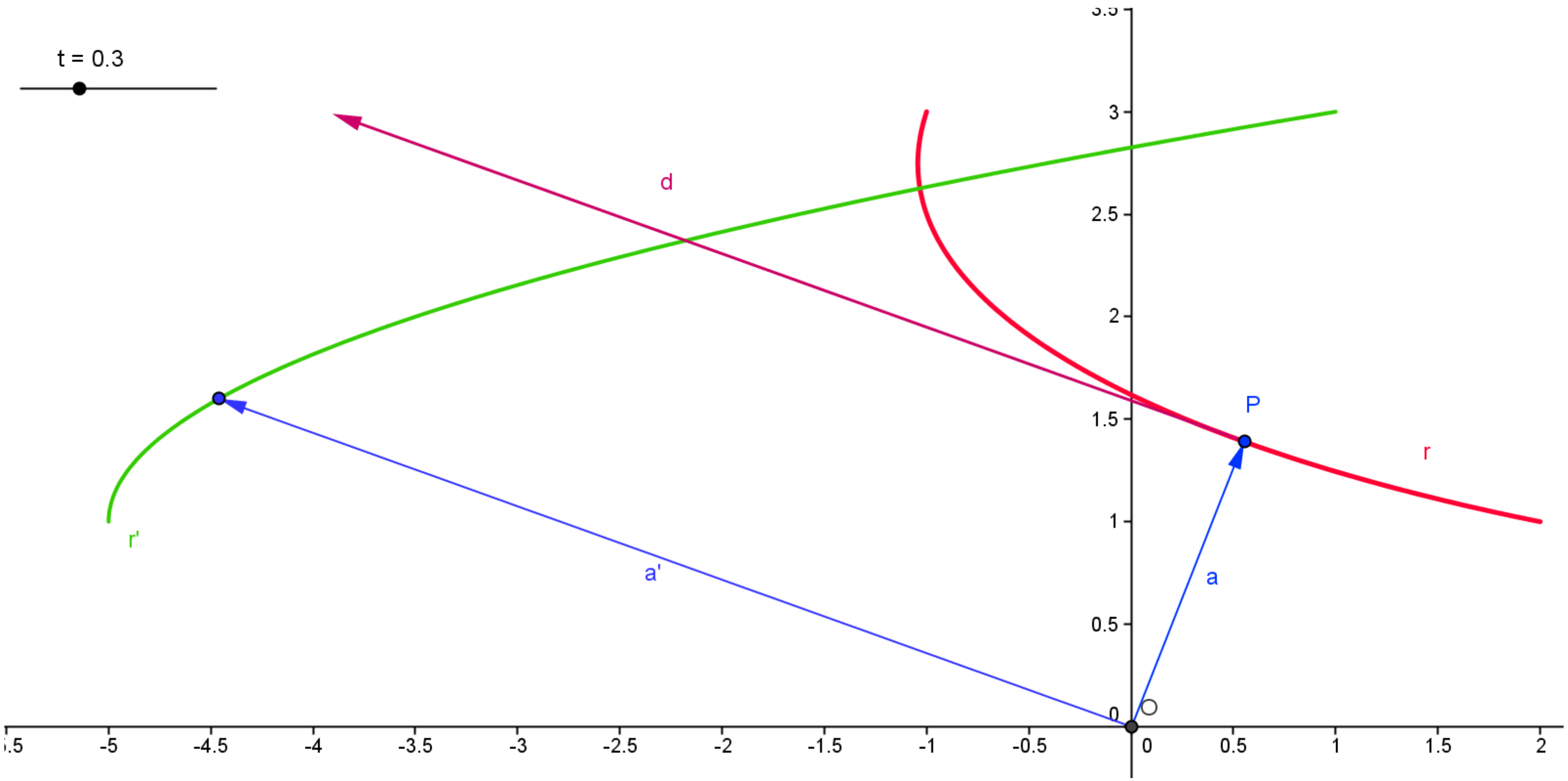
$$\mathbf{r}(t) = (x(t), y(t))$$

derivative $\mathbf{r}'(t_0) = (x'(t_0), y'(t_0))$

is direction vector of tangent line to the curve $k$,

which is graph of vector function $\mathbf{r}(t)$ at the point

$$X = [x(t_0), y(t_0)] \in \boldsymbol{E}^2$$




Planar curve **r**($t$) = $(2t^3\text{-}5t+2,\ t^2+t+1)$ (in red)

Derivative **r**′($t$) = $(6t^2\text{-}5,\ 2t+1)$ is also a planar curve (in green)

For three-dimensional vector function defined on $I \subset \boldsymbol{R}$

$\mathbf{r}(t) = (x(t), y(t), z(t))$

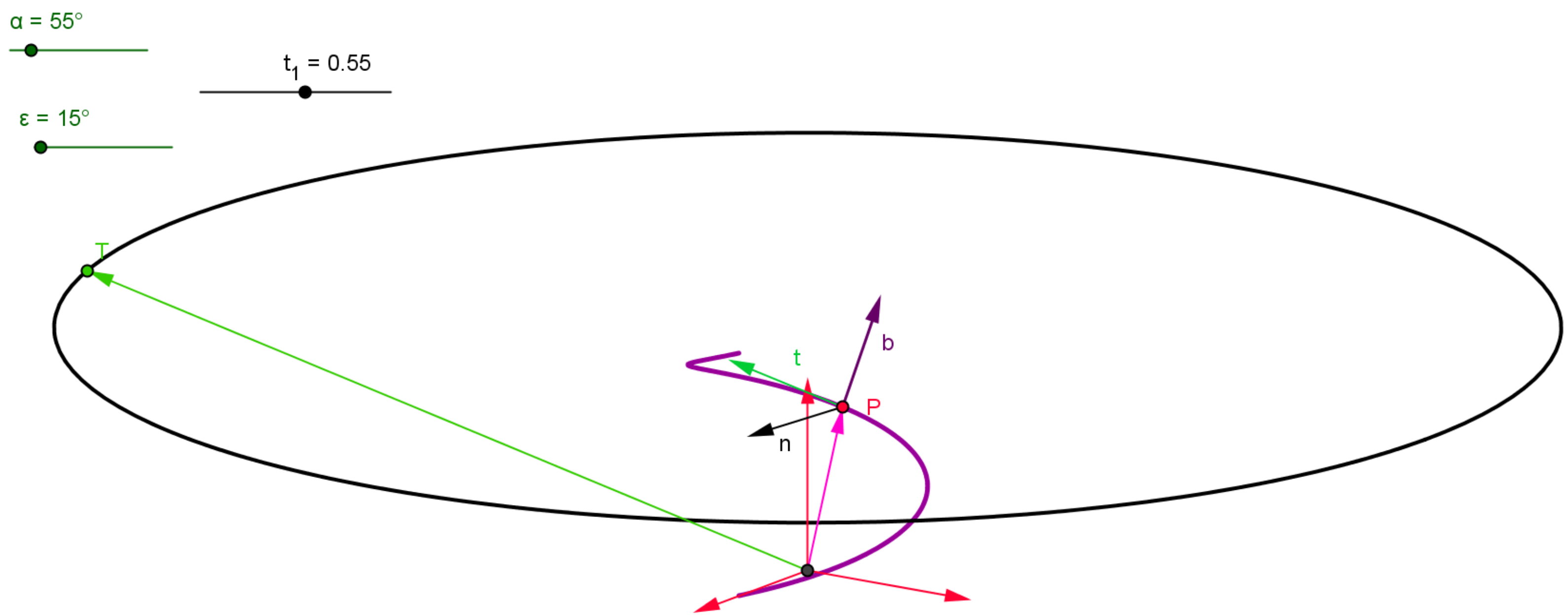
derivative $\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$

is direction vector of the tangent line to the curve $K$,

which is graph of vector function $\mathbf{r}(t)$,

at the point

$X = [x(t_0), y(t_0), z(t_0)] \in \boldsymbol{E}^3$




$\mathbf{r}(t) = (a\cos(t), b\sin(t), vt)$, $t \in <0, 2\pi>$ graph is elliptic helix,
for $a = b$ cylindrical helix

$\mathbf{r}'(t) = (-a\sin(t), b\cos(t), v)$, $t \in <0, 2\pi>$ graph is ellipse in plane $z = v$
for $a = b$ circle