Vector functions

Vector space

Set of all vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, with components that are real numbers $a_i \in \boldsymbol{R}$ is called a vector space over real numbers and it is denoted $\boldsymbol{V}^n(\boldsymbol{R})$, if the following properties hold:

1. for any real number k and any vector $\mathbf{a} \in \boldsymbol{V}^n(\boldsymbol{R})$ holds $k.\mathbf{a} \in \boldsymbol{V}^n(\boldsymbol{R})$
2. for any 2 vectors $\mathbf{a}, \mathbf{b} \in \boldsymbol{V}^n(\boldsymbol{R})$ holds $\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, $\mathbf{c} \in \boldsymbol{V}^n(\boldsymbol{R})$
3. for any 2 vectors $\mathbf{a}, \mathbf{b} \in \boldsymbol{V}^n(\boldsymbol{R})$ and 2 real numbers k, l holds

    $k(\mathbf{a} + \mathbf{b}) = k.\mathbf{a} + k.\mathbf{b}$, $(k + l).\mathbf{a} = k.\mathbf{a} + l.\mathbf{a}$

    $(k.l).\mathbf{a} = k.(l.\mathbf{a})$, $1.\ \mathbf{a} = \mathbf{a}$, $\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}$

We define additional operations in the vector space $\boldsymbol{V}^n(\boldsymbol{R})$:

4. scalar product of vectors $$\mathbf{a}.\mathbf{b} = \sum_{i=1}^{n} a_i . b_i$$

5. vector product of two vectors , exclusively for $V^3(\boldsymbol{R})$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

6. mixed scalar product of three vectors, exclusively for $V^3(\boldsymbol{R})$

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7. norm – size (length) of vectors $$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

Let M be an interval in real numbers and $\boldsymbol{V}^n(\boldsymbol{R})$ be n-dimensional vector space over real numbers $\boldsymbol{R}$.

Mapping **f** from $\boldsymbol{R}$ to vector space $\boldsymbol{V}^n(\boldsymbol{R})$, in which any real number $t \in M$ is attached a vector from space $\boldsymbol{V}^n(\boldsymbol{R})$

$$\mathbf{f}: M \to \boldsymbol{V}^n(\boldsymbol{R}), \quad t \to \mathbf{f}(t)$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are real functions (scalar coordinate functions) of one real variable t defined on the same interval M,
will be called vector function of one real variable t.

Interval M is called domain of definition of vector function **f** - $D(\mathbf{f})$.

Image of point $t_0 \in M$ in mapping $\mathbf{f}$ is vector, related to number t_0, and denoted

$$\mathbf{f}(t_0) = (f_1(t_0), f_2(t_0), \ldots, f_n(t_0))$$

which is called the value of function $\mathbf{f}$ at the point t_0. Range of function $\mathbf{f}$ is set

$$H(\mathbf{f}) = \{\mathbf{a} \in \boldsymbol{V}^n(R); \exists t \in D(\mathbf{f}), \mathbf{a} = \mathbf{f}(t)\}$$

For $n = 1$ we receive function with one real variable t, $\mathbf{f}(t) = (f_1(t))$.

For $n = 2$ we receive function $\mathbf{f}(t) = (f_1(t), f_2(t))$ of real variable t, which can be denoted also as

$$\mathbf{f}(t) = (x(t), y(t)) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

For $n = 3$ function is $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$ that can be also denoted

$$\mathbf{f}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Vector function $\mathbf{f}$ is uniquely determined by its domain of definition $D(\mathbf{f}) \subset \boldsymbol{R}$ and by formulas (scalar coordinate functions), according to which any point $t \in D(\mathbf{f})$ can be attached just one vector $\mathbf{f}(t)$ called value of the vector function.

Function can be represented in different ways:

- by words
- by table of function values
- by graph
- analytically – by means of mathematical formula or equation

Let **f** be a vector function of one variable t defined on interval M in real numbers.

Graph (hodograph) of function **f** is set $G(\mathbf{f})$ of all such points

$X = [x_1, x_2, ..., x_n] \in \boldsymbol{E}^n$, for which:

1. $\mathbf{OX} = \mathbf{a} = (x_1, x_2, ..., x_n) \in \boldsymbol{V}_n$

2. $\exists t \in \boldsymbol{R}; \mathbf{a} = \mathbf{f}(t) = (f_1(t), f_2(t), ..., f_n(t))$

Geometrically we can naturally interpret only graphs of vector functions for $n = 1, 2$ and 3.

Graph of vector function $\mathbf{f}(t) = (x(t), y(t))$ is a plane curve.

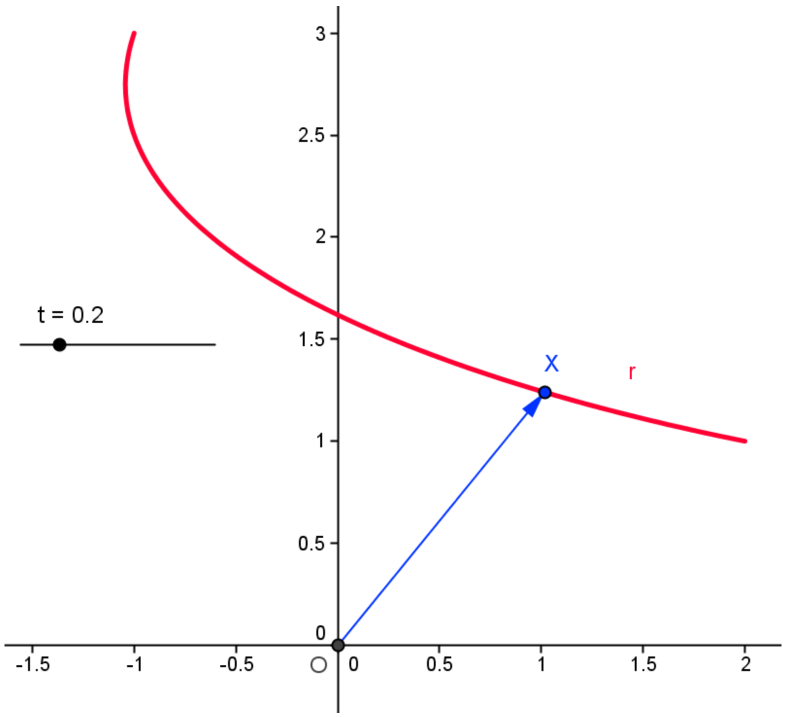
Graph of vector function $\mathbf{f}(t) = (x(t), y(t), z(t))$ is a space curve.

Graph of vector function $\mathbf{r}(t) = (x(t), y(t))$ defined on $I \subset \boldsymbol{R}$

is a plane curve k, i.e. set of all such points $X = [x, y] \in \boldsymbol{E}^2$, if:

1. $\mathbf{OX} = \mathbf{a} = (xa, ya) \in V^2(\boldsymbol{R})$, vector $\mathbf{a}$ is called position vector of a point on the curve

2. $\exists t \in I; \mathbf{a} = \mathbf{r}(t) = (x(t), y(t))$



Parametric equations of curve

$x = x(t),\ y = y(t),\ t \in I$

Vector equation

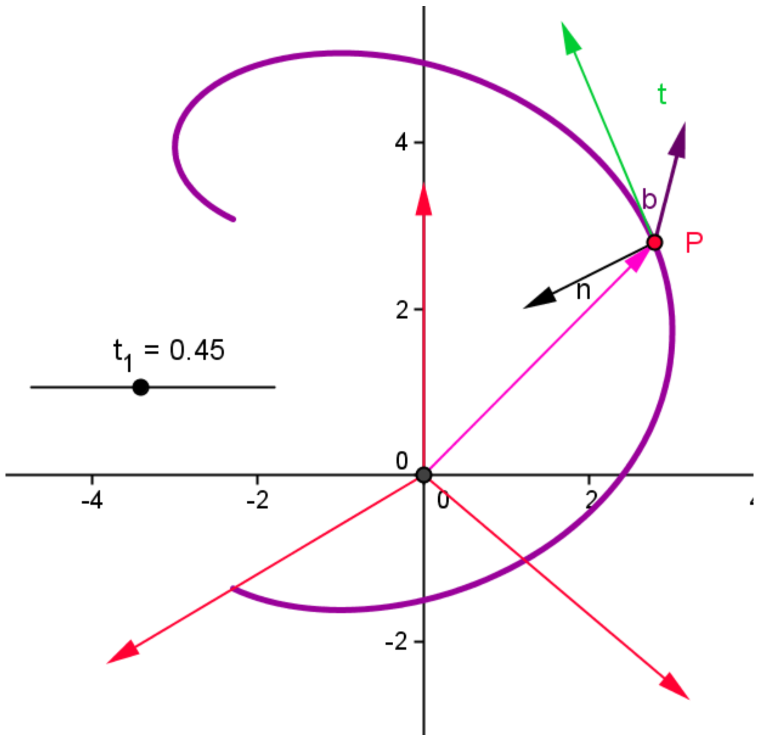
$\mathbf{r}(t) = (x(t), y(t)),\ t \in I$

Circle

$\mathbf{r}(t) = (r\cos(t), r\sin(t)),\ t \in \langle 0, 2\pi \rangle,\ r \in \boldsymbol{R}$

Graph of vector function **r**(t) = (x(t), y(t), z(t)) defined on $I \subset \boldsymbol{R}$ is a space curve K, i.e. set of all such points $X = [x, y, z] \in \boldsymbol{E}^3$, for which:

1. **OX** = **a** = (xa, ya, za) $\in V^3(\boldsymbol{R})$, vector **a** is called position vector of a point on the curve



2. $\exists t \in \boldsymbol{I}; \mathbf{a} = \mathbf{r}(t) = (x(t), y(t), z(t))$

Parametric equations of curve

$x = x(t),\ y = y(t),\ z = z(t),\ t \in I$

Vector equation

$\mathbf{r}(t) = (x(t), y(t), z(t)),\ t \in I$

Helix

$\mathbf{r}(t) = (r\cos(t), r\sin(t), vt),\ t \in \langle 0, 2\pi \rangle,\ r \in \boldsymbol{R}$

Concept of vector function of one variable can be generalised to the vector function of more variables.
Let M be a reular simply connected region – subset of $\boldsymbol{R}^n$
and $V^n(\boldsymbol{R})$ be vector space over real numbers.

Mapping **f** from set $\boldsymbol{R}^n$ to vector space $V^n(\boldsymbol{R})$, in which any n-tuple of real numbers $(t_1, t_2, ..., t_n) \in M$ is attached a vector from space $V^n(\boldsymbol{R})$

$$\mathbf{f}: M \to V^n(\boldsymbol{R}), \ (t_1, t_2, ..., t_n) \to \mathbf{f}(t_1, t_2, ..., t_n)$$

$$\mathbf{f}(t_1, t_2, ..., t_n) = (f_1(t_1, t_2, ..., t_n), f_2(t_1, t_2, ..., t_n), ..., f_n(t_1, t_2, ..., t_n))$$

where $f_1(t_1, t_2, ..., t_n), f_2(t_1, t_2, ..., t_n), ..., f_n(t_1, t_2, ..., t_n)$ are real functions with n real variables $t_1, t_2, ..., t_n$ (coordinate functions) defined on the same region M, is a vector function $\mathbf{f}(t_1, t_2, ..., t_n)$ of n real variables $t_1, t_2, ..., t_n$.

Vector function of two variables defined on $M \subset \boldsymbol{R}^2$, while range is subset of space $V^2(\boldsymbol{R})$ is a mapping, in which any ordered pair of real numbers $(u, v) \in M$ is attched a vector from $\boldsymbol{V}^2(\boldsymbol{R})$

$$\mathbf{f}: M \to \boldsymbol{V}^2(\boldsymbol{R}), \quad (u, v) \to \mathbf{f}(u, v)$$

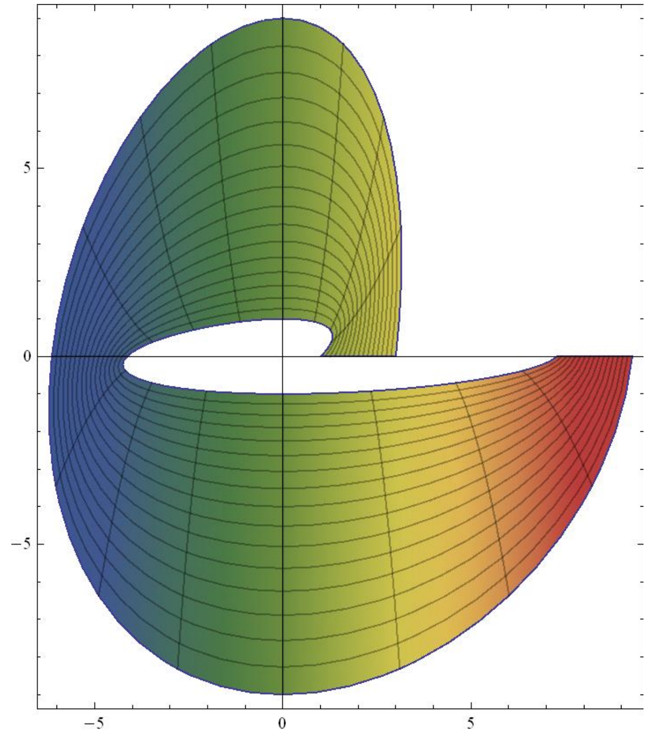
$$\mathbf{f}(u, v) = (x(u, v), y(u, v))$$

Graph of function $\mathbf{f}$ is a planar region in the space $\boldsymbol{E}^2$.

Planar region

$\mathbf{f}(u, v) = ((\mathrm{u} + \mathrm{v})\cos v, u^2 \sin v)$

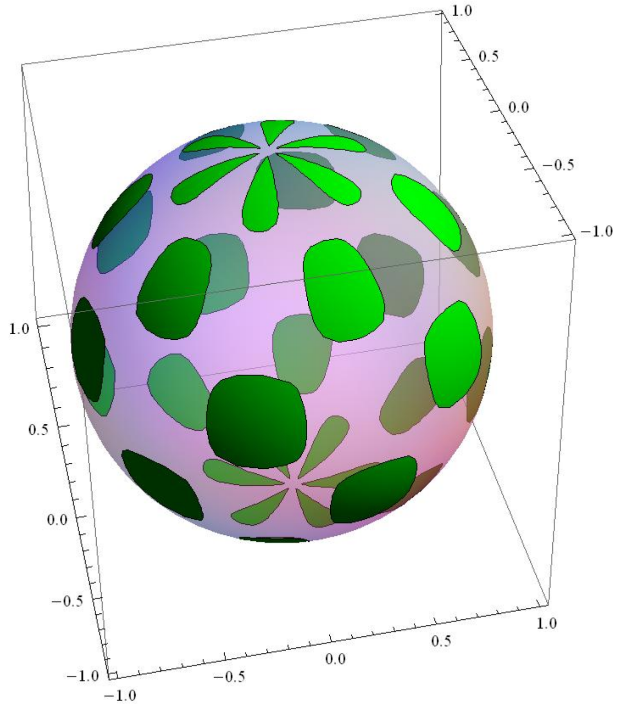
$(u, v) \in <1, 3> \times <0, 2\pi>$



Vector function of two variables defined on $M \subset \boldsymbol{R}^2$ with the range of values in the vector space $V^3(\boldsymbol{R})$ is mapping, in which any ordered pair of real numbers $(u, v) \in M$ is attached a vector from $V^3(\boldsymbol{R})$

$$\mathbf{f}: M \rightarrow \boldsymbol{V}^3(\boldsymbol{R}),\ (u, v) \rightarrow \mathbf{f}(u, v)$$

$$\mathbf{f}(u, v) = (x(u, v), y(u, v), z(u, v))$$



Graph of function is a surface in space $\boldsymbol{E}^3$, which can be mapped in 2D view by means of one from the known projection methods used in descriptive geometry.

Sphere

$\mathbf{p}(u, v) = (r\cos u \cos v, r\cos u \sin v, r\sin v)$

$(u, v) \in \langle 0, 2\pi \rangle^2, r \in \boldsymbol{R}$