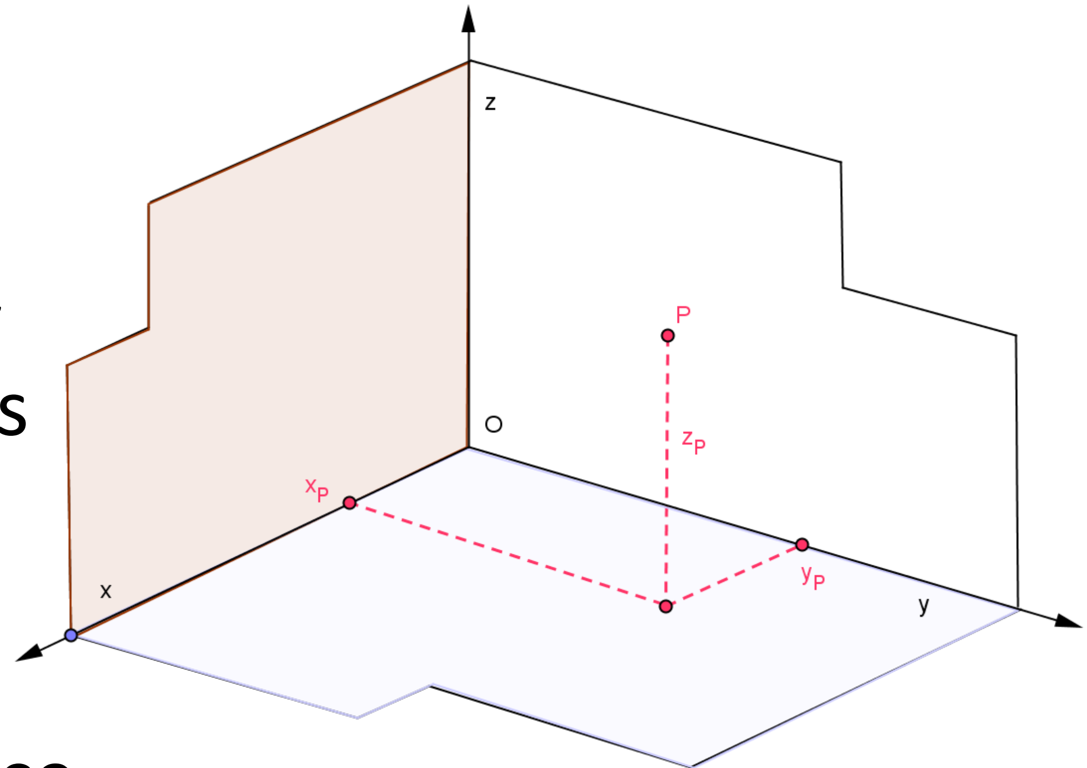
Vectors and operations with vectors

Orthogonal Cartesian coordinate system

- origin O
- 3 coordinate axes

  $x = o_x,\ y = o_y,\ z = o_z$
- 3 coordinate planes

  $Oxy,\ Oxz,\ Oyz$



Each point in the space

is identified with an order triple of real numbers

Cartesian coordinates $P = [x_P,\ y_P,\ z_P]$

Euclidean distance of two points

$P_1 = [x_1, y_1, z_1]$ and $P_2 = [x_2, y_2, z_2]$

is determined by formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vector in E^3 is determined by 2 points

$$\mathbf{a} = \overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Position vector of point P is vector

$$\vec{\mathbf{a}} = \overrightarrow{OP} = (x_P, y_P, z_P)$$

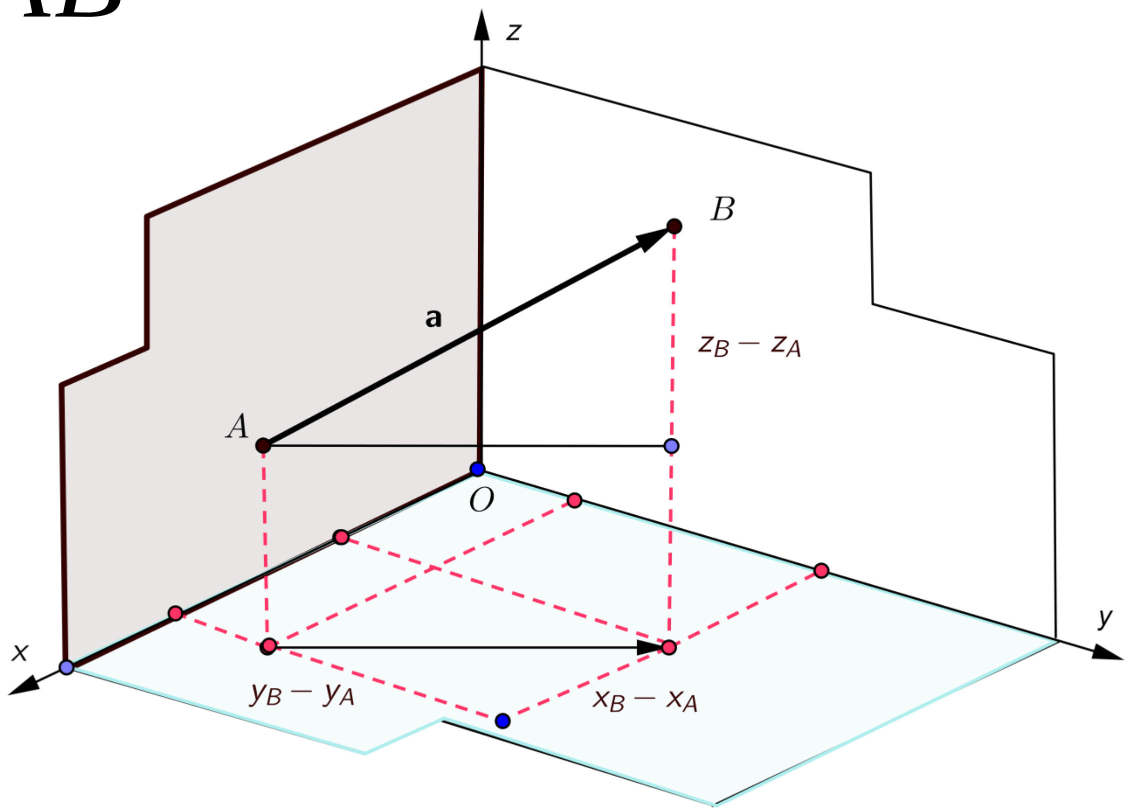
Vector is any oriented line segment

$$\vec{\mathbf{a}} = \overrightarrow{AB}$$

A – start point

B – end point

of vector location



Vector coordinates

$$A = [x_A, y_A, z_A]$$

$$B = [x_B, y_B, z_B]$$

$$\vec{\mathbf{a}} = B - A = (x_B - x_A, y_B - y_A, z_B - z_A)$$

Norm – length of vector

$$\vec{\mathbf{v}} = \overrightarrow{AB} = B - A = (x_B - x_A, y_B - y_A, z_B - z_A) = (v_1, v_2, v_3)$$

is the distance of its determining points *A* and *B*

$$\left|\vec{\mathbf{v}}\right| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

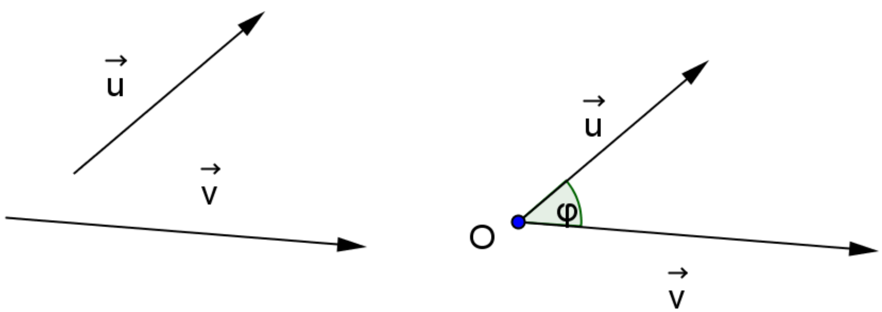
$\left|\vec{\mathbf{v}}\right| = 1$ - unit vector

$\vec{\mathbf{0}} = (0,0,0)$ - zero vector

Angle of 2 vectors $\quad \vec{\mathbf{u}}, \vec{\mathbf{v}} \quad \varphi = \angle(\vec{\mathbf{u}}, \vec{\mathbf{v}})$

$$\left|\vec{\mathbf{u}}\right| \neq 0, \left|\vec{\mathbf{v}}\right| \neq 0, \varphi \in \langle 0, \pi \rangle$$

non-parallel vectors



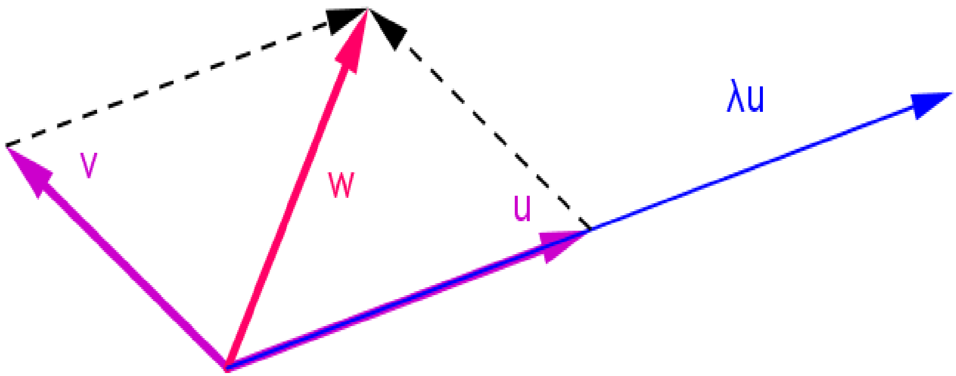
$$\varphi \in (0, \pi)$$

$\varphi = 0$ $\qquad\qquad$ $\varphi = \pi$

parallel (collinear) vectors

Operations with vectors

Sum of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$



$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{w}} = \vec{\mathbf{u}} + \vec{\mathbf{v}} =$$

$$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

Scalar multiple of a vector

$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{u}} \,||\, \vec{\mathbf{v}} \Leftrightarrow \exists \lambda \in R, \vec{\mathbf{u}} = \lambda \vec{\mathbf{v}}$$

$$\lambda \vec{\mathbf{u}} = (\lambda u_1, \lambda u_2, \lambda u_3)$$

Vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are linearly independent, if there exists no such linear combination of these vectors

$$k.\vec{\mathbf{a}} + l.\vec{\mathbf{b}} + m.\vec{\mathbf{c}} = \vec{\mathbf{0}}$$

that at least one from coefficients *k*, *l*, *m* is a nonzero real number, i.e. $k^2 + l^2 + m^2 \neq 0$.

Vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are linearly dependent, if at least one of them is a linear combination of the two others.

Unit vectors $\vec{\mathbf{i}} = (1,0,0), \vec{\mathbf{j}} = (0,1,0), \vec{\mathbf{k}} = (0,0,1)$ form ortho-normal basis of the three dimensional space.

Any vector in the space can be represented as linear combination of the vectors from the basis, while its coefficients are coordinates of vector in this basis.

Scalar product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$

$$\vec{\mathbf{u}} \,.\, \vec{\mathbf{v}} = \left|\vec{\mathbf{u}}\right| . \left|\vec{\mathbf{v}}\right| \cos\varphi, \qquad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right)$$

$$\text{If } \vec{\mathbf{u}} = \vec{\mathbf{0}} \vee \vec{\mathbf{v}} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{u}} \,.\, \vec{\mathbf{v}} = 0$$

$$\vec{\mathbf{u}} = (u_1, u_2, u_3),\ \vec{\mathbf{v}} = (v_1, v_2, v_3),\ \vec{\mathbf{u}} \,.\, \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\cos\varphi = \frac{\vec{\mathbf{u}} \,.\, \vec{\mathbf{v}}}{\left|\vec{\mathbf{u}}\right| . \left|\vec{\mathbf{v}}\right|}, \qquad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right), \qquad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

$$\vec{\mathbf{u}} \perp \vec{\mathbf{v}} \Leftrightarrow \vec{\mathbf{u}} \,.\, \vec{\mathbf{v}} = 0, \qquad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Vector product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ is vector $\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$

1. $\left|\vec{\mathbf{w}}\right| = \left|\vec{\mathbf{u}}\right| . \left|\vec{\mathbf{v}}\right| \sin\varphi, \quad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right)$

2. $\vec{\mathbf{w}} \perp \vec{\mathbf{u}} \wedge \vec{\mathbf{w}} \perp \vec{\mathbf{v}}$

3. Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ form right-handed system.

$$\vec{\mathbf{u}} = (u_1, u_2, u_3) \qquad \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{u}} \,||\, \vec{\mathbf{v}} \Leftrightarrow \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

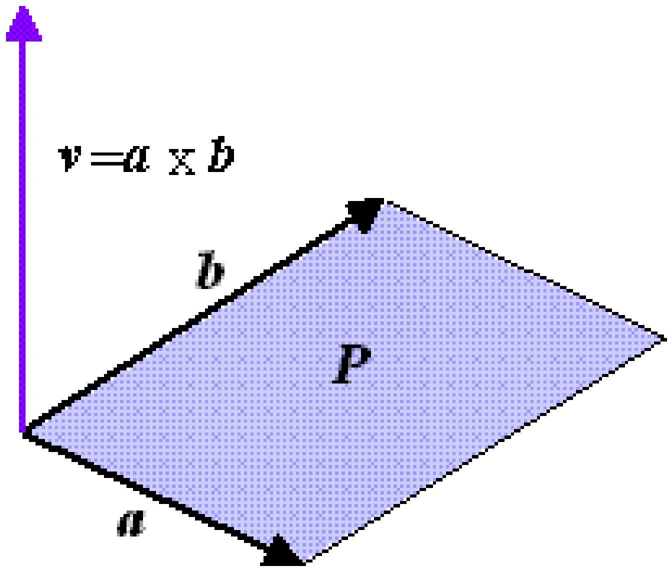
Geometric interpretation of vector product

Length of vector $\vec{\mathbf{v}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$

equals to the area *P* of a parallelogram with sides

in non-collinear vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}$

$$P = \left| \vec{\mathbf{a}} \times \vec{\mathbf{b}} \right|$$



Mixed triple product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ is number

$$\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = \left(\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right).\vec{\mathbf{w}}$$

$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{w}} = (w_1, w_2, w_3)$$

$$\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ are in one plane (are coplanar)

if and only if $\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = 0$

Geometric interpretation of mixed product

Absolute value of mixed scalar product of 3 vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ equals to the volume *V* of parallelepiped with edges in these non-coplanar vectors .

$$V = \left| \left[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \right] \right|$$