Surface integrals

Let S be a regular surface given by parameterization

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k},\ (u, v) \in G \subset \boldsymbol{R}^2$$

and $f(X)$ be a scalar function defined and continuous on S.

Surface integral of the first type is integral from function f

$$\iint_S f(X)\,dS = \iint_G f(\boldsymbol{r}(u,v)).\left|\boldsymbol{r}'_{\boldsymbol{u}}(u,v)\times\boldsymbol{r}'_{\boldsymbol{v}}(u,v)\right|du\,dv$$

Integral does not depend on the chosen parameterization of surface S.

Coordinate form of the formula is

$$\iint_S f(x,y,z)\,dS =$$

$$= \iint_G f(x(u,v), y(u,v), z(u,v)).\sqrt{\begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}^2 + \begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix}^2 + \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}^2}\,dudv$$

If surface S is graph of function $z = z(x, y)$, $[x, y] \in G \subset \boldsymbol{R}^2$, integral has form

$$\iint_S f(x,y,z)\,dS =$$

$$= \iint_G f(x,y,z(x,y)).\sqrt{1+[z'_x(x,y)]^2+[z'_y(x,y)]^2}\,dx\,dy$$

Physical meaning - mass of surface S with density function $f(X)$.

Geometric interpretation

Area of surface S

$$\iint_S dS$$

Let a regular oriented surface S be given by parameterization

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k},\ (u, v) \in G \subset \boldsymbol{R}^2$$

and vector function $\boldsymbol{F}(X)$ be defined and continuous on S

$$\boldsymbol{F}(X) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$

Surface integral of the second type is integral from function $\boldsymbol{F}$

$$\iint_S \boldsymbol{F}(X)\,d\boldsymbol{S} = \pm \iint_G \boldsymbol{F}(\boldsymbol{r}(u,v)).(\boldsymbol{r}'_{\boldsymbol{u}}(u,v) \times \boldsymbol{r}'_{\boldsymbol{v}}(u,v))\,du\,dv$$

Integral does not depend on the chosen parameterization of surface S, but it depends only on its orientation.

Coordinate form the formula is

$$\iint_S \boldsymbol{F}(x,y,z)d\boldsymbol{S} = \pm\iint_S \begin{vmatrix} P(\boldsymbol{r}(u,v)) & Q(\boldsymbol{r}(u,v)) & R(\boldsymbol{r}(u,v)) \\ x'_u(u,v) & y'_u(u,v) & z'_u(u,v) \\ x'_v(u,v) & y'_v(u,v) & z'_v(u,v) \end{vmatrix} du\,dv$$

If surface S is graph of function $z = z(x, y)$, $[x, y] \in G \subset \boldsymbol{R}^2$, integral has form

$$\iint_S \boldsymbol{F}(x,y,z)d\boldsymbol{S} =$$

$$= \pm\iint_S [-P(x,y,z(x,y)).z'_x(x,y) - Q(x,y,z(x,y)).z'_y(x,y) + R(x,y,z(x,y))]\,dx\,dy$$

Physical meaning

- calculation of a flow in vector field determined by vector function ***F***, while surface S is located in the given field
- if function ***F*** determines velocity of flowing liquid (gass), then

$$\int_S \boldsymbol{F}(x, y, z)dS$$

represents the total quantity of liquid (gass), that has flown through the surface S in directon of orientation for a time unit (surface flow rate)

Gauss – Ostrogradsky theorem

Let V be a three-dimensional region bounded by a simple closed surface S oriented by exterior normal vector. Let $\boldsymbol{F}$ be a vector function defined and continuous on region V with continuous partial derivatives of the first order. Then it holds

$$\iint_S \boldsymbol{F}(x,y,z)\,d\boldsymbol{S} = \iiint_V div\,\boldsymbol{F}\,dx\,dy\,dz$$

Consequence

If region V satisfies conditions of the Gauss-Ostrogradsky theorem, then the volume of region V can be calculated as follows

$$\mu(V) = \frac{1}{3}\iint_S (x,y,z)\,d\boldsymbol{S}$$

Stokes´ theorem

Let *S* be smooth oriented surface bounded by closed smooth curve *K* and let vector function ***F*** and its first partial derivatives be continuous on certain open set containing surface *S*. Then, if orientation of both surface *S* and curve *K* is conformable, it holds

$$\oint_K \boldsymbol{F}(x,y,z)d\boldsymbol{r} = \iint_S rot\boldsymbol{F}(x,y,z)d\boldsymbol{S}$$

If for an oriented surface *S* and a simple curve *K* located on surface *S* holds that from the direction of the unit normal vector we turn round the curve *K* in an anti-clock-wise manner, we say that curve *K* and surface *S* are conformably oriented.