Line (curve) integrals

Let $f(X)$ be a scalar function, non-negative and continuous on region $\Omega \subset \boldsymbol{R}^2$ and let a regular plane curve K be given by parameterization

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j},\ t \in \langle \alpha, \beta \rangle \subset \boldsymbol{R}.$$

Line (curve) integral of the first type is integral from function f

$$\int_K f(X)\,d\boldsymbol{r} = \int_\alpha^\beta f(x(t), y(t)).\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Integral does not depend on the chosen parameterization of curve K.

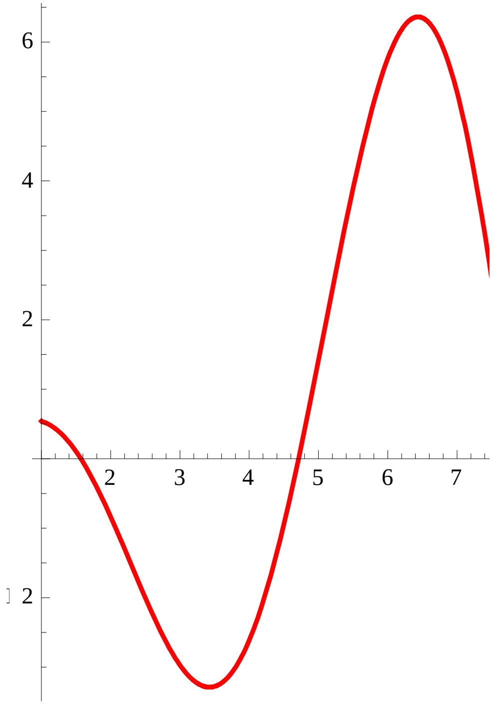
Vector form of the formula is

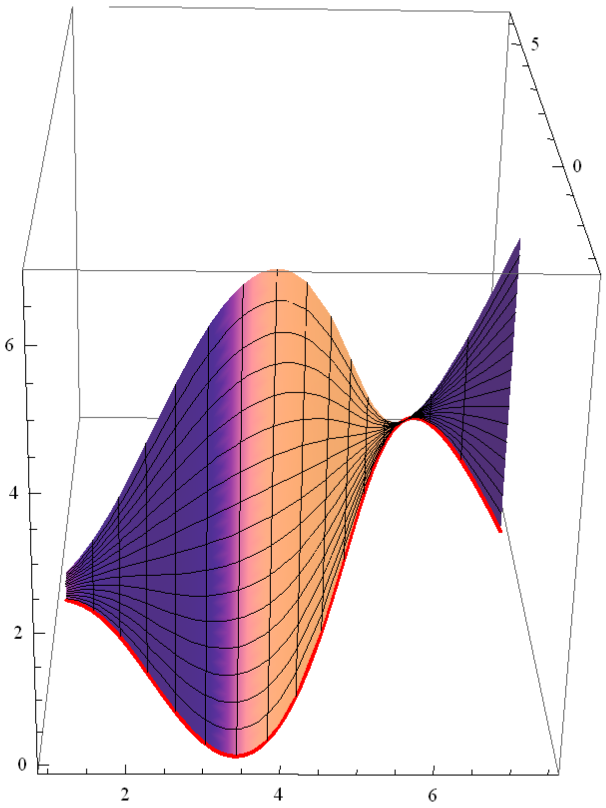
$$\int_K f(X)\,d\boldsymbol{r} = \int_\alpha^\beta f(\boldsymbol{r}(t)).\left|\boldsymbol{r}'(t)\right|dt$$

Physical meaning - mass of a plane curve K with density function $f(X)$.

Geometric interpretation

Area of a patch of ruled surface (cylindrical patch) whose generating lines are in direction of axis z and basic curve is plane curve K. Surface patch is bounded from above by values of function $f(x, y)$ at the points of basic curve K.





Length of curve K is

$$l = \int_K d\boldsymbol{r} = \int_\alpha^\beta |\boldsymbol{r}'(t)| dt$$

Let $f(X)$ be a scalar function, non-negative and continuous on region $\Omega \subset \boldsymbol{R}^3$ and let a regular curve K be given by parameterization

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k},\ t \in \langle \alpha, \beta \rangle \subset \boldsymbol{R}$$

Line (curve) integral of the first type from function f is

$$\int_K f(X)\,d\boldsymbol{r} = \int_\alpha^\beta f(x(t), y(t), z(t)).\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Integral does not depend on the chosen parameterization of curve K.

Vector form of the formula is

$$\int_K f(X)\,d\boldsymbol{r} = \int_\alpha^\beta f(\boldsymbol{r}(t)).\left|\boldsymbol{r}'(t)\right|dt$$

Physical meaning – mass of a space curve K with density function $f(X)$.

Let vector function $\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$ be differentiable on region $\Omega \subset \boldsymbol{R}^2$ and let a regular oriented curve K be given by parameterization

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j},\ t \in \langle \alpha, \beta \rangle \subset \boldsymbol{R}$$

Line (curve) integral of the second type is integral from function $\boldsymbol{F}$

$$\int_K P\,dx + Q\,dy = \int_\alpha^\beta \left[P\frac{dx}{dt} + Q\frac{dy}{dt} \right] dt$$

Vector form of the formula is

$$\int_K \boldsymbol{F}\,d\boldsymbol{r} = \pm\int_\alpha^\beta \boldsymbol{F}(\boldsymbol{r}(t)).\boldsymbol{r}'(t)\,dt = \int_\alpha^\beta [P(x(t), y(t)).x'(t) + Q(x(t), y(t)).y'(t)]dt$$

Let vector function be given

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\,\boldsymbol{i} + Q(x, y, z)\,\boldsymbol{j} + R(x, y, z)\,\boldsymbol{j}$$

differentiable on region $\Omega \subset \boldsymbol{R}^3$ and let a regular oriented curve K be given by parameterization $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k},\ t \in \langle \alpha, \beta \rangle \subset \boldsymbol{R}.$

Line (curve) integral of the second type from function $\boldsymbol{F}$ is

$$\int_K P\,dx + Q\,dy + R\,dz = \int_\alpha^\beta \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right] dt$$

Vector and coordinate form of the formula is

$$\int_K \boldsymbol{F}\,d\boldsymbol{r} = \pm \int_\alpha^\beta \boldsymbol{F}(\boldsymbol{r}(t)).\boldsymbol{r}'(t)\,dt =$$

$$= \int_\alpha^\beta [P(x(t), y(t), z(t)).x'(t) + Q(x(t), y(t), z(t)).y'(t) + R(x(t), y(t), z(t)).y'(t)]dt$$

Let $(\Omega, \boldsymbol{F})$ be a vector field.

We say that line (curve) integral from function $\boldsymbol{F}$ does not depend on the integration path (trajectory) in the region Ω, if for any two oriented, piece-wise smooth curves K_1, K_2 that are located in the region Ω and both possess common start and end points holds

$$\int_{K_1} \boldsymbol{F}\, d\boldsymbol{r} = \int_{K_2} \boldsymbol{F}\, d\boldsymbol{r}$$

Line (curve) integral from function $\boldsymbol{F}$ does not depend on the integration path in the region Ω if and only if the line (curve) integral from function $\boldsymbol{F}$ on an arbitrary closed, piece-wise smooth curve is vanishing (is equal to zero).

Line (curve) integral does not depend on the integration path if and only if the vector field (Ω, $\boldsymbol{F}$) is a potential field.

If U is a potential of the feld
$$\boldsymbol{F} = grad\,U = \frac{\partial U}{\partial x}\boldsymbol{i} + \frac{\partial U}{\partial y}\boldsymbol{j} + \frac{\partial U}{\partial z}\boldsymbol{k}$$

then for an arbitrary piece-wise smooth curve K in Ω with the start point A and end point B holds
$$\int_K \boldsymbol{F}\, d\boldsymbol{r} = U(B) - U(A)$$

Let continuous partial derivatives of the first order of function $\boldsymbol{F}$ exist on Ω .

Necessary condition for the field (Ω, $\boldsymbol{F}$) to be a potential field is the validity of equation rot $\boldsymbol{F} = \boldsymbol{0}$ at any point of the region Ω.

If Ω is a simply continuous region (with boundary in a simple closed piece-wise curve), this condition is also a sufficent condition.

Green´s theorem

Let G be a simply connected region with boundary in a simple closed piece-wise curve K. Let continuous partial derivatives with respect to x and y exist on G for vector function

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}.$$

Then it holds

$$\oint_K \boldsymbol{F}\,.d\boldsymbol{r} = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Consequence

Let G be a plane region bounded by positively oriented closed simple and piece-wise smooth curve K.

Then, for the area of this region holds $\quad P = \dfrac{1}{2} \oint_K x\,dy - y\,dx$