

Series with variable terms (function series)

Sequence consisting of terms that are all in the form of a function is called a sequence with variable terms, or a function sequence

$$\{f_n\} = \{f_n(x)\} = f_1(x), f_2(x), \dots, f_n(x), \dots$$

Let all functions that are terms of the sequence $\{f_n\}$ be defined at some point (number) x_0 , then the convergence (divergence) of the given function sequence $\{f_n\}$ at the point x_0 can be considered.

Sequence $\{f_n\}$ that is convergent (divergent) at all points of some set M is said to be convergent (divergent) at the set M .

Set of all such points at which the given sequence $\{f_n\}$ is convergent is called the **domain of convergence** of the sequence $\{f_n\}$.

Let the sequence $\{f_n\}$ be convergent on the set M , then a function is defined on the set M such that

$$f(x_0) = \lim_{n \rightarrow \infty} f_n(x_0), \forall x_0 \in M$$

and it is called the limit of the sequence $\{f_n\}$ on the set M

$$f = \lim_{n \rightarrow \infty} f_n$$

Therefore

$$f = \lim_{n \rightarrow \infty} f_n \text{ on } M \Leftrightarrow \forall x_0 \in M, \forall \varepsilon > 0, \exists n_0 : \forall n \in N, n > n_0 \Rightarrow |f_n(x_0) - f(x_0)| < \varepsilon$$

Function f need not to be continuous on M , and if it is continuous we speak about uniform convergence of the sequence with variable terms.

Function series

Let all terms of a sequence with variable terms $\{f_n\}$ be defined on a non-empty set M . Expression

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + \dots + f_n + \dots$$

is called the series with variable terms (function series).

Let

$$s_n = \sum_{i=1}^n f_i = f_1 + f_2 + \dots + f_n, n = 1, 2, \dots$$

Function s_n is called the n -th partial sum of the series $\sum_{n=1}^{\infty} f_n$.

Function series is called convergent (divergent) on the set M , if the sequence of its partial sums $\{s_n\}$ is convergent (divergent) on M .

Domain of convergence of a function series is the domain of convergence of the related function sequence $\{s_n\}$.

$$s = \lim_{n \rightarrow \infty} s_n$$

If there exists the proper limit

on the set M , then function s is called the sum of the series on M ,

$$\sum_{n=1}^{\infty} f_n = s, \quad \sum_{n=1}^{\infty} f_n(x) = s(x)$$

If point x_0 is in the domain of convergence of the function series, then the

number series $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent and its sum is the value of function

s at the point x_0 , $\sum_{n=1}^{\infty} f_n(x_0) = s(x_0)$.

How to find domain of convergence of a function series and its sum?

Weierstrass criterion

Let the number series $\sum_{n=1}^{\infty} a_n$ be convergent and let the following relation

holds for all terms of the function series $\sum_{n=1}^{\infty} f_n$

$$|f_n(x)| \leq a_n, \forall x \in M,$$

Then the function series $\sum_{n=1}^{\infty} f_n$ is convergent on the set M .

Power series

Function series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + \dots + a_n(x-a)^n + \dots$$

involving positive powers of a variable x and constant numbers a_n is called the power series in x with the centre at the point a . Numbers a_n are coefficients of the power series.

For $a = 0$ we get

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \dots + a_n x^n + \dots$$

Power series is a certain generalisation of the concept of polynomial, and it has several similar properties. Power series is uniquely determined by the coefficients a_n and the centre a .

Domain of convergence of a power series is an interval, or one point, alternatively.

For any power series there exists number $R \geq 0$ (possibly $R = \infty$) such, that the respective number series is absolutely convergent at all points $x \in (a - R, a + R)$, and it is divergent at all points $x \notin (a - R, a + R)$.

R is the **radius of convergence** and interval $I = (a - R, a + R)$ is the **interval of convergence** of the power series.

Series convergent only at its centre a has $R = 0$, $I = \{a\}$.

If $R = \infty$, series is convergent on the set of all real numbers, $I = (-\infty, \infty)$.

Power series can be convergent also at the boundary points of the interval I .

Let the coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ be such that there exists a limit

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

or limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \rho$$

and let $\rho \in (0, \infty)$, then the radius of convergence of the power series is

$$R = \frac{1}{\rho}$$

If $\rho = 0$, then $R = \infty$, and if $\rho = \infty$, then $R = 0$.

Let $R > 0$ be the radius of convergence of a power series with centre a , and let $s(x)$ be the series sum, while $I = (a - R, a + R)$ be the series domain of convergence. Then the following statements hold:

- function $s(x)$ is continuous on the interval I

- if $\langle b, c \rangle \subset I$, then there exists
$$\int_b^c s(x) dx = \sum_{n=0}^{\infty} \int_b^c a_n (x - a)^n dx$$

which can be derived by integration of the power series by separate terms

- function $s(x)$ has antiderivative on interval I , which can be derived by integration of power series by separate terms
- function $s(x)$ has derivative on interval I , which can be derived by differentiation of power series by separate terms
- integration and differentiation of the power series does not change its domain of convergence.

Taylor series

Let function f be differentiable on interval $J = (a - c, a + c)$ up to the order $n + 1$.

Then for any $x \in J$ there exists such ξ between x and a , that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Polynomial of the degree n

$$T_n(f, a, x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n -th Taylor polynomial of function f with the centre at the

point a , and formula $R_n(f, a, x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$

is the reminder of the n -th Taylor polynomial.

It holds

$$f(a) = T_n(f, a, a)$$

Value of function f at the point a equals to the value of the Taylor polynomial $T_n(f, a, x)$ at the point a .

$$T'_n(f, a, a) = f'(a), \dots, T_n^{(n)}(f, a, a) = f^{(n)}(a)$$

Values of the derivatives of function f at the point a up to the order n are equal to the values of the respective derivatives of the Taylor polynomial $T_n(f, a, x)$ at the point a .

With increasing n the value of the remainder $R_n(f, a, x)$ is decreasing.

Let there exist derivatives of function f at the point a of all orders, then a power series with the centre at the point a can be constructed (as generalisation of the Taylor polynomial)

Taylor series of function f at the point a $T(f, a, x)$

$$T(f, a, x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

For $a = 0$ the series is called **Maclaurin series**.

Sum of Taylor series $T_n(f, a, x)$ need not be always function f , but this is the case in majority of examples.

There exist also such functions, whose Taylor series are convergent, but not to these particular functions.

Necessary and sufficient condition for the Taylor series of a function to be convergent to the sum in this function

Let function f has derivatives of all orders on interval $J = (a - c, a + c)$.
Then for all $x \in J$ holds

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

if and only if $\lim_{n \rightarrow \infty} R_n(f, a, x) = 0$.

Sufficient condition for the Taylor series of a function to be convergent to the sum in this function

Let there exist derivatives of all orders of function f on interval $J = (a - c, a + c)$. If there exists constant $K > 0$ such, that

$$\forall n \in N, \forall x \in J : |f^{(n)}(x)| < K$$

then for all $x \in J$ holds

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Let function f be equal to the sum of its Taylor series on a certain interval, then it is said to be expandable on this interval to the power series and this series is called (the infinite) expansion of function f .

A function can be expanded to a power series with the centre a in a unique way, and there exists always a Taylor series of the given function with the centre at the point a .

$$\sin x = T(\sin x, 0, x) = \frac{x}{1!} - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots, x \in (-\infty, \infty)$$

$$\cos x = T(\cos x, 0, x) = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots, x \in (-\infty, \infty)$$

$$e^x = T(e^x, 0, x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} \dots + \frac{x^n}{n!} + \dots, x \in (-\infty, \infty)$$

$$\ln(x+1) = T(\ln(x+1), 0, x) = x - \frac{x^2}{2} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots, x \in (-1, 1)$$

$$\arctan x = T(\arctan x, 0, x) = x - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots, x \in \langle -1, 1 \rangle$$

$$(x+1)^p = T((x+1)^p, 0, x) = 1 + \binom{p}{1}x + \dots + \binom{p}{n}x^n + \dots, x \in (-1, 1)$$