Fourier series

Series with variable terms

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called trigonometric series with period 2π.

Numbers a_0, a_1, b_1, ..., a_n, b_n, ... are coefficients of this series.

If a series is convergent on interval $\langle -\pi, \pi\rangle$ and it holds

$$s(x) = \frac{1}{2}a_0 + \sum_{i=1}^{n}(a_n \cos nx + b_n \sin nx)$$

then function s is also periodic with period 2π, as sum of infinitely many periodic functions with period 2π.

- if $a_n = 0$, $\forall\, n = 0, 1, 2, ...$, series is called sinusoidal, function s is odd
- if $b_n = 0$, $\forall\, n = 1, 2, ...$, series is called cosinusoidal, function s is even.

Let f be periodic function with period 2π, and let coefficients a_0, a_1, b_1, ..., a_n, b_n of trigonometric series satisfy the following

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, n = 1,2,...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, n = 1,2,...$$

Then trigonometric series with such coefficients (called Fourier coefficients) is known as Fourier series of function f.

If a Fourier series is determined for function f, under what assumptions it is convergent to function f?

When does exist an expansion of function f to the Fourier series?

Function f is said to be piece-wise monotonneous on interval $\langle a, b \rangle$, if this interval can be decomposed to finitely many sub-intervals such, that function f is monotonneous on each of them.

If a function is piece-wise monotonneous and bounded on certain interval, then only points of discontinuty of the first type can exist on this interval. One-sided limits of respective function exist at these points.

If function is periodic and its behaviour is repeated on given interval, the above property is valid on entire domain of definition of function.

Sufficient condition for the expansion of function to Fourier series

If a periodic function *f* with period 2π is piece-wise monotonneous and bounded on interval ⟨-π, π⟩, then Fourier series of function *f* converges to this function at all points of continuity and at all points of discontinuity it converges to the arithmetic average of one-sided limits at this points.

If on interval ⟨-π, π⟩ holds

$$s(x) = \frac{1}{2}a_0 + \sum_{i=1}^{n}(a_n \cos nx + b_n \sin nx)$$

then for arbitrary real number x_0 holds

$$s(x_0) = \frac{1}{2}\left(\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right)$$

If function f is continuous at point x_0, then $s(x_0) = f(x_0)$.

Consequence

Fourier series of a periodic continuous function with period 2π converges to this function on interval $(-\infty, \infty)$.

If a Fourier series converges to a given function on some set of real numbers, then this function is said to be expandable to the Fourier series on given set, and this series is called expansion of given function to the Fourier series.

Fourier series of even and odd functions

If a periodic function *f* with period 2π is an even function, then it holds

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx, n = 1,2,...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0, n = 1,2,...$$

Fourier series contains only even functions, it is called cosinusoidal series and it has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

If a periodic function *f* with period 2π is odd, then it holds

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0, n = 1,2,\ldots$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx, n = 1,2,\ldots$$

Fourier series contains only odd functions, it is called sinusoidal series and it has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier series of functions with general period

Let a periodic function *f* with period $T = 2l$ be piece-wise monotonneous and bounded on interval $\langle -l, l\rangle$. Then its Fourier series converges to function *f* at all points of continuity, and at the points of discontinuity it converges to the arithmetic average of one-sided limits at this points. Fourier series has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

and for Fourier coefficients it holds

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, n = 1,2,...$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx, n = 1,2,...$$

Then following statements hold for a function f with period $2l$:

Fourier series of an even function f is a cosinusoidal series.

Fourier series of an odd function f is a sinusoidal series.