PROBLEMS ON SURFACE INTEGRALS

Problem 1: Evaluate the following surface integrals of the first type:

a) $\iint_S dS$, S is surface patch $x = u\cos v,\ y = u\sin v,\ z = 4v,\ 0 \le u \le 3,\ 0 \le v \le \pi$

b) $\iint_S z\,dS$, S is patch of helicoid $\boldsymbol{r} = u\cos v\,\mathbf{i} + u\sin v\,\boldsymbol{j} + v\,\boldsymbol{k}$, $(u, v) \in \langle 0, a\rangle \times \langle 0, 2\pi\rangle$

c) $\iint_S z^2 dS$, S is patch of conical surface

$\boldsymbol{r} = u\cos v\sin\alpha\,\mathbf{i} + u\sin v\sin\alpha\,\boldsymbol{j} + u\cos\alpha\,\boldsymbol{k}$, $(u, v) \in \langle 0, a\rangle \times \langle 0, 2\pi\rangle$, $\alpha \in (0, \pi/2)$

d) $\iint_S xyz\,dS$, S is part of plane $x + y + z = 1$, located in the first octant

e) $\iint_S (x^2 + y^2)\,dS$, S is patch of conical surface $x^2 + y^2 = z^2,\ 2 \le z \le 3$

f) $\iint_S |xyz|\,dS$, S is patch of paraboloid $z = x^2 + y^2,\ z \le 1$

g) $\iint_S \frac{\sqrt{x^2 + y^2}}{z}\,dS$, S is patch of paraboloid $z = x^2 + y^2$ bounded by plane $z = a$

h) $\iint_S \frac{1}{\sqrt{x^2 + y^2}}\,dS$, S is patch of hyperbolic paraboloid $z = xy$ $z = x^2 + y^2$ bounded by cylindrical surface $x^2 + y^2 = R^2, |z| \le R$

i) $\iint_S (x^2 + y^2)\,dS$, S is sphere $x^2 + y^2 + z^2 = a^2$

j) $\iint_S (x^2 + y^2)z\,dS$, S is upper hemisphere with centre at origin and radius R

k) $\iint_S x^2 y^2\,dS$, S is spherical patch $x^2 + y^2 + z^2 = R^2, z \ge 0$

Solution: a) $r'_u \times r'_v = (4\sin v, -4\cos v, u),\ I = \int_0^3 \left(\int_0^\pi \sqrt{16 + u^2}\,dv \right) du = \frac{\pi}{2}(15 + 16\ln 2)$

b) $I = \pi^2 (a\sqrt{a^2 + 1} + \ln(a + \sqrt{a^2 + 1}))$

c) $I = \frac{\pi a^4}{2} \sin\alpha \cos^2\alpha$

d) $x = u, y = v, z = 1 - u - v, 0 \le u \le 1, 0 \le v \le 1 - u, I = \int_0^1 \left(\int_0^{1-u} uv(1 - u - v)\sqrt{3}\,dv \right) du = \frac{\sqrt{3}}{120}$

e) $x = u\cos v, y = u\sin v, z = u, 2 \le u \le 3, 0 \le v \le 2\pi, I = \int_0^{2\pi} \left(\int_2^3 u^2 u\sqrt{2}\,du \right) dv = \frac{65\sqrt{2}\pi}{2}$

f) $I = \frac{125\sqrt{5} - 1}{420}$

g) $x = u\cos v, y = u\sin v, z = u^2, 0 \le u \le \sqrt{a}, 0 \le v \le 2\pi, \mathbf{r}'_u \times \mathbf{r}'_v = (-2u^2\cos v, -2u^2\sin v, u)$
$|\mathbf{r}'_u \times \mathbf{r}'| = u\sqrt{4u^2+1}, I = \int_0^{\sqrt{a}}\left(\int_0^{2\pi}\sqrt{1+4u^2}\,dv\right)du = \pi\left[\sqrt{a}\sqrt{1+4a}+\frac{1}{2}\ln\left(2\sqrt{a}+\sqrt{1+4a}\right)\right]$

h) $I = \pi\left[R\sqrt{R^2+1}+\ln\left(R+\sqrt{R^2+1}\right)\right]$

i) $x = a\cos u\cos v, y = a\sin u\cos v, z = a\sin v, 0 \le u \le 2\pi, -\frac{\pi}{2} \le v \le \frac{\pi}{2},$
$\mathbf{r}'_u \times \mathbf{r}'_v = (a^2\cos u\cos^2 v, a^2\sin u\cos^2 v, a^2\sin v\cos v), |\mathbf{r}'_u \times \mathbf{r}'_v| = a^2\cos v,$
$I = \int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2 v a^2\cos v\,dv\right)du = \frac{8}{3}\pi a^4$

j) $x = R\cos u\cos v, y = R\sin u\cos v, z = R\sin v, 0 \le u \le 2\pi, 0 \le v \le \frac{\pi}{2},$
$\mathbf{r}'_u \times \mathbf{r}'_v = (R^2\cos u\cos^2 v, R\sin u\cos^2 v, R^2\sin v\cos v), |\mathbf{r}'_u \times \mathbf{r}'_v| = R^2\cos v,$
$I = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}} R^2\cos^2 vR\sin vR^2\cos v\,dv\right)du = \frac{\pi}{2}R^5$

k) $x = R\cos u\cos v, y = R\sin u\cos v, z = R\sin v, 0 \le u \le 2\pi, 0 \le v \le \frac{\pi}{2},$
$I = R^6\int_0^{2\pi}\cos^2 u\sin^2 u\,du\int_0^{\frac{\pi}{2}}\cos^4 v\cos v\,dv = \frac{2\pi}{15}R^6$

Problem 2: Evaluate the following surface integrals of the second type:

a) $\iint_S xdydz + ydxdz + zdxdy$, S is surface patch $\boldsymbol{r} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + (uv+1)\boldsymbol{k}$, $(u, v) \in\langle 0.\ 1\rangle^2$ oriented so that normal vectors form acute angles to vector $\boldsymbol{k}$

b) $\iint_S ydydz + xdxdz + zdxdy$, S is surface patch $\boldsymbol{r} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + (uv+1)\boldsymbol{k}$, $(u, v) \in\langle 0.\ 1\rangle^2$ oriented so that normal vectors form acute angles to vector $\boldsymbol{k}$

c) $\iint_S xdydz + ydxdz + zdxdy$, S is patch of paraboloid $\boldsymbol{r} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + (u^2 + v^2)\boldsymbol{k}$, $(u^2 + v^2) \le 1$ oriented so that normal vectors form obtuse angles to vector $\boldsymbol{k}$

d) $\iint_S zdxdy + xdxdz + ydydz$, S is surface patch $x - y + z - 1 = 0$, $x \ge 0$, $y \le 0$, $z \ge 0$, oriented so that normal vectors form acute angles to vector $\boldsymbol{k}$

e) $\iint_S ydydz + zdxdz + xdxdy$, S is part of plane $x + y + z = a$, located in the first octant oriented so that normal vector form acute angle to vector $\boldsymbol{k}$

f) $\iint_S xzdydx + xydydz + yzdxdz$, S is surface of pyramid bounded by planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, oriented by outer normal

g) $\iint_S xzdydx + xydydz + yzdxdz$, S is surface of cube $\langle 0,\, 1\rangle^3$ oriented by outer normal

h) $\iint_S xzdydz + ydxdz + zdxdy$, S is surface of cylinder with base in plane $z = 0$, centre in the origin of coordinate system, radius R and height H, oriented by outer normal

i) $\iint_S x^2dydz + z^2dxdy$, S is patch of conical surface $x^2 + y^2 = z^2$, $0 \le z \le 1$, oriented so that normal vectors form obtuse angles to vector $\boldsymbol{k}$

j) $\iint_S xdydz + ydxdz + zdxdy$, S is sphere $x^2 + y^2 + z^2 = a^2$ oriented by outer normal

k) $\iint_S \frac{dydz}{x} + \frac{dxdz}{y} + \frac{dxdy}{z}$, S is ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ oriented by outer normal

Solutions: a) $I = \int_0^1 \left(\int_0^1 (1 - uv)du \right) dv = \frac{3}{4}$

b) $I = \int_0^1 \left(\int_0^1 (-u^2 - v^2 + uv + 1)du \right) dv = \frac{7}{12}$

c) $I = \iint_G (u^2 + v^2)dudv = \int_0^1 \left(\int_0^{2\pi} \rho^2 . \rho \, d\varphi \right) d\rho = \frac{\pi}{2}$

$x = u, y = v, z = 1 - u + v, 0 \le u \le 1, u - 1 \le v \le 0, \boldsymbol{r}'_u \times \boldsymbol{r}'_v = (1, -1, 1)$

d) $I = \int_0^1 \left(\int_{u-1}^0 (1 - 2u - 2v)\, dv \right) du = -\frac{1}{6}$

e) $x = u, y = v, z = a - u - v, 0 \le u \le a, 0 \le v \le a - u, I = \frac{a^3}{2}$

f) $I = \frac{1}{8}$

$S_1 : x = u, y = v, z = 0, 0 \le u \le 1, 0 \le v \le 1, I_1 = -\int_0^1 \int_0^1 0\, dx\, dy = 0$

$S_2 : x = u, y = v, z = 1, 0 \le u \le 1, 0 \le v \le 1, I_2 = \int_0^1 \int_0^1 1\, dx\, dy = 1,$

$S_3 : x = u, y = 0, z = v, 0 \le u \le 1, 0 \le v \le 1, I_3 = \int_0^1 \int_0^1 0\, dx\, dz = 0,$

g) $S_4 : x = u, y = 1, z = v, 0 \le u \le 1, 0 \le v \le 1, I_4 = \int_0^1 \int_0^1 1\, dx\, dz = 1,$

$S_5 : x = 0, y = u, z = v, 0 \le u \le 1, 0 \le v \le 1, I_5 = \int_0^1 \int_0^1 0\, dy\, dz = 0,$

$S_5 : x = 1, y = u, z = v, 0 \le u \le 1, 0 \le v \le 1, I_5 = \int_0^1 \int_0^1 1\, dy\, dz = 1,$

$I = 3$

h) $$\begin{aligned}&S_1: x=R\cos u, y=R\sin u, z=v, 0\le u\le 2\pi, 0\le v\le H, I_1=2\pi HR^2\\&S_2: x=u\cos v, y=u\sin v, z=0, 0\le u\le R, 0\le v\le 2\pi, I_2=0,\\&S_3: x=u\cos v, y=u\sin v, z=H, 0\le u\le R, 0\le v\le 2\pi, I_3=\pi HR^2,\\&I=3\pi HR^2\end{aligned}$$

i) $$\begin{aligned}&x=u\cos v, y=u\sin v, z=u, 0\le u\le 1, 0\le v\le 2\pi, \mathbf{r}'_u\times\mathbf{r}'_v=(-u\cos v,-u\sin v,u)\\&I=\int_0^1\left(\int_0^{2\pi}(u^3\cos^3 v-u^3)\,dv\right)du=-\frac{\pi}{2}\end{aligned}$$

j) $$\begin{aligned}&x=a\cos u\cos v, y=a\sin u\cos v, z=a\sin v, 0\le u\le 2\pi, -\frac{\pi}{2}\le v\le\frac{\pi}{2},\\&I=a^3\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos v\,dv\right)du=4\pi a^3\end{aligned}$$

k) $$\begin{aligned}&x=a\cos u\cos v, y=b\sin u\cos v, z=c\sin v, 0\le u\le 2\pi, -\frac{\pi}{2}\le v\le\frac{\pi}{2},\\&\mathbf{r}'_u\times\mathbf{r}'_v=(bc\cos u\cos^2 v, ac\sin u\cos^2 v, ab\sin v\cos v)\\&I=\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right)\cos v\,dv\right)du=4\pi\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right)\end{aligned}$$

Problem 3: Evaluate surface integrals by means of Gauss-Ostrogradsky theorem:

a) $\iint_S x^2dydz+y^2dxdz+z^2dxdy$, S is surface of cube $\langle 0, 1\rangle^3$, oriented by outer normal

b) $\iint_S xzdxdy+xydydz+yzdxdz$, S is surface of pyramid bounded by planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$, oriented by outer normal

c) $\iint_S x^3dydz+y^3dxdz+z^3dxdy$, S is sphere $x^2 + y^2 + z^2 = a^2$, oriented by outer normal

d) find volume of ellipsoid with centre in origin and semiaxes a, b, c in the coordinate axes x, y, z

Solutions: a) $I=2\int_0^1\int_0^1\int_0^1(x+y+z)\,dx\,dy\,dz=3$

b) $I=\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dz\,dy\,dx=\frac{1}{8}$

c) $I=\frac{12}{5}\pi a^5$

d) $$\begin{aligned}&x=a\cos u\cos v, y=b\sin u\cos v, z=c\sin v, 0\le u\le 2\pi, -\frac{\pi}{2}\le v\le\frac{\pi}{2},\\&P=\frac{2\pi abc}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos v\,dv=\frac{4\pi}{3}abc\end{aligned}$$

Problem 4: Evaluate the following integrals by means of Stokes´ theorem:

a) $\iint_K (y+z)dx + (z+x)dy + (x+y)dz$, K is ellipse $x = a\sin^2 t$, $y = 2a\sin t \cos t$, $z = a\cos^2 t$, $t \in \langle 0, 2\pi \rangle$, oriented conformably with the given parametric representation

b) $\iint_K xzdxdy + xydydz + yzdxdz$, K is intersection curve of cylindrical surface $x^2+y^2= 2y$ and plane $y = z$

c) $\iint_K (y^2+z^2)dx + (x^2+z^2)dy + (x^2+y^2)dz$, K is intersection curve of two spheres $x^2 + y^2 + z^2 = 2bx$, $x^2 + y^2 = 2ax$, $z \geq 0$, $0 < a < b$, oriented conformably with the smaller patch of the sphere S: $x^2 + y^2 + z^2 = 2bx$ whose boundary it is, while surface S is oriented by its outer normal

d) $\iint_K (x^2 - yz)dx + (x^2 - xz)dy + (z^2 - xy)dz$, K is arc AB of helix with parametric representation $x = a \cos t$, $y = a \sin t$, $z = ht/2\pi$, $A = [a, 0, 0]$, $B = [a, 0, h]$, while curve K is closed by line segmet BA

Solutions: a) rot $\boldsymbol{F} = \boldsymbol{0}$, I = 0

b) rot $\boldsymbol{F} = \boldsymbol{0}$, I = 0

c)
$$x = u, y = v, z = \sqrt{2bu - u^2 - v^2}, 0 \leq u \leq 2a, -\sqrt{2au - u^2} \leq v \leq \sqrt{2au - u^2},$$
$$rot\boldsymbol{F} = \left(2(y-z), 2(z-x), 2(x-y)\right), I = 2b\iint_S \left(\frac{v}{\sqrt{2bu - u^2 - v^2}} + 1\right) dv\, du = 2\pi a^2 b$$

d) $rot\boldsymbol{F} = \boldsymbol{0}, I = \int_0^1 h^3 t^2 dt = \dfrac{h^3}{3}$