PROBLEMS ON LINE (CURVE) INTEGRALS

Problem 1: Evaluate the following integrals of the first type:

a) $\int_K \sqrt{2y}\,d\boldsymbol{r}$, where K is a part of cycloid $\boldsymbol{r}(t) = (a(t - \sin t), a(1 - \cos t))$, $t \in \langle 0, 2\pi\rangle$

b) $\int_K (x^2 + y^2)\,d\boldsymbol{r}$, where K is curve $\boldsymbol{r}(t) = a(\cos t + t \sin t, \sin t - t \cos t))$, $t \in \langle 0, 2\pi\rangle$

c) $\int_K \frac{z^2}{x^2 + y^2}\,d\boldsymbol{r}$, where K is an arc of helix $\boldsymbol{r}(t) = a(\cos t, \sin t, t))$, $t \in \langle 0, 2\pi\rangle$

d) $\int_K \frac{1}{\sqrt{x^2 + y^2}}\,d\boldsymbol{r}$, where K is line segment $x - 2y - 4 = 0$, $0 \le x \le 4$

e) $\int_K \frac{1}{x - y}\,d\boldsymbol{r}$, where K is line segment AB, $A = [0, -2]$, $B = [4, 0]$

f) $\int_K y\,d\boldsymbol{r}$, where K is parabolic arc $y = 2\sqrt{x}, 0 \le x \le 1$

g) $\int_K x^2\,d\boldsymbol{r}$, where K is curve $y = \ln x, 1 \le x \le 2$

h) $\int_K x\,d\boldsymbol{r}$, where K is parabolic arc $y = x^2, 1 \le x \le 2$

i) $\int_K x^2 y\,d\boldsymbol{r}$, where K is circular arc $x^2 + y^2 = a^2$ located in the first quadrant

j) $\int_K xy\,d\boldsymbol{r}$, where K is elliptic arc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ located in the first quadrant

Solution: a) $I = 2a\sqrt{a}\int_0^{2\pi}(1 - \cos t)dt = 4\pi a\sqrt{a}$

b) $I = 2\pi^2 a^3(1 + 2\pi^2)$

c) $I = a\sqrt{2}\int_0^{2\pi} t^2 dt = \frac{8\sqrt{2}a\pi^3}{3}$

d) $I = \int_{-\frac{1}{2}}^{2} \frac{1}{\sqrt{t^2 + 1}}dt = \ln\frac{3\sqrt{5} + 7}{2}$

e) $x = 4t, y = -2 + 2t, 0 \le t \le 1, I = \sqrt{5}\ln 2$

f) $x = t, y = 2\sqrt{t}, 0 \le t \le 1, I = 2\int_0^1 \sqrt{t + 1}dt = \frac{4}{3}(2\sqrt{2} - 1)$

g) $x = t, y = \ln t, 1 \le t \le 2, I = \frac{1}{3}(5\sqrt{5} - 2\sqrt{2})$

h) $x = t, y = t^2, 1 \le t \le 2, I = \frac{1}{12}(17\sqrt{17} - 5\sqrt{5})$

i) $x = a\cos t, y = a\sin t, t \in \langle 0, 2\pi\rangle, I = \frac{a^4}{3}$

j) $x = a\cos t, y = b\sin t, 0 \le t \le \frac{\pi}{2},$

$$I = \int_0^{\frac{\pi}{2}} ab\sin t\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt = \frac{ab(a^2 + ab + b^2)}{3(a + b)}$$

Problem 2: Evaluate the following integrals of the second type:

a) $\int_K y^2 dx$, K is closed curve, composed from a part of ellipse $\boldsymbol{r}(t) = (a \cos t, b \sin t)$, $t \in \langle 0, \pi \rangle$ and line segment $y = 0$, $x \in \langle -a, a \rangle$

b) $\int_K \frac{x}{y} dx + \frac{y}{y-a} dy$, K is a part of cycloid $\boldsymbol{r}(t) = (a(t - \sin t), a(1 - \cos t))$, $t \in \langle \pi/6, \pi/3 \rangle$

c) $\int_K \frac{x^2 dy - y^2 dx}{x^{\frac{5}{3}} + y^{\frac{5}{3}}}$, K is part of asteroid $\boldsymbol{r}(t) = (a \cos^3 t, a \sin^3 t))$, $t \in \langle 0, \pi/2 \rangle$

d) $\int_K x\, dx + y\, dy + (xz - y)\, dz$, K is $\boldsymbol{r}(t) = (t^2, 2t, 4t^3)$, from $A = [0,0,0]$ to $B = [1,2,4]$

e) $\int_K yz\, dx + xz\, dy + xy\, dz$, K is helical arc $\boldsymbol{r}(t) = (a \cos t, a \sin t, bt/2\pi)$, $0 \le t \le 2\pi$

f) $\int_K 4x \sin^2 y\, dx + y \cos^2 2x\, dy$, K is line segment AB, $A = [0, 0]$, $B = [3, 6]$

g) $\int_K x\, dy - y\, dx$, K is curve $y = x^3$, $0 \le x \le 2$

h) $\int_K xy\, dx$, K is part of sinusoidal curve $y = \sin x$, $0 \le x \le \pi$

i) $\int_K (x - y)\, dx + (x + y)\, dy$, K is parabolic arc $x = y^2$, $0 \le x \le 2$

j) $\int_K \frac{(x+y)\, dx - (x-y)\, dy}{x^2 + y^2}$, K is positively oriented circle $x^2 + y^2 = a^2$

k) $\int_K y^2 dx + z^2 dy + x^2 dz$, K is part of Viviani curve $x^2 + y^2 + z^2 = a^2$, $x^2 + y^2 = ax$, $z \ge 0$, points $A = [a,0,0], B = \left[\frac{a}{2}, \frac{a}{2}, \frac{a}{\sqrt{2}}\right], C = [0,0,a]$ determine curve orientation

Solution: a) $I_1 = -\frac{4}{3}ab^2, K_2 : x = -a + 2at, y = 0, 0 \le t \le 1, I_2 = 0 \Rightarrow I = -\frac{4}{3}ab^2$

b) $I = a\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (t - gt)\, dt = a\left(\frac{\pi^2}{24} + \ln\frac{1}{\sqrt{3}}\right)$

c) $I = \frac{3}{8} a^{\frac{4}{3}} \int_0^{\frac{\pi}{2}} (t - \cos 4t)\, dt = \frac{3\pi}{16} a^{\frac{4}{3}}$

d) $I = 5/8$

e) $I = 0$

f) $K : x = 3t, y = 6t, 0 \le t \le 1, I = 18$

g) $I = 8$

h) $I = \pi$

i) $K : x = t^2, y = t, -\sqrt{2} \le t \le \sqrt{2}, I = -\frac{4\sqrt{2}}{3}$

j) $K : x = a\cos t, y = a\sin t, 0 \le t \le 2\pi, I = -2\pi$

k) $K : x = \frac{a}{2}(1 + \cos t), y = \frac{a}{2}\sin t, z = a\sin\frac{t}{2}, 0 \le t \le 2\pi, I = -\frac{\pi a^3}{4}$

Problem 3: Find, whether given integrals depend or not on the integration path:

a) $\int_K (2x+3y)\,dx+(3x-4y)\,dy$

b) $\int_K (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$

c) $\int_K (x^2-2yz)\,dx+(y^2-2xz)\,dy+(x^2-2xy)dz$

d) $\int_K \left(3x^2+\frac{y}{(x+z)^2}\right)dx+\frac{dy}{x+z}+\frac{y}{(x+z)^2}dz$

e) $\int_K \left(\frac{1}{x^2}+\frac{3y^2}{x^4}\right)dx-\frac{2y}{x^3}dy$

Solution: a) Not

b) Not

c) Not

d) Not

e) Not, if curve K does not intersect axis z.

Problem 4: Evaluate the following integrals by means of Green´s theorem:

a) $\int_K y^2\,dx+x\,dy$, K is boundary of square bounded by lines x=1, x=-1, y=1, y=-1

b) $\int_K y^2\,dx+x\,dy$, K is positively oriented circle with radius 2 and centre O = [0,0]

c) $\int_K (x+y)\,dx-(x-y)\,dy$ K is positively oriented ellipse $\boldsymbol{r}(t)$=(a cos t, b sin t), $t\in\langle 0,2\pi\rangle$

Solution: a) 4 b) 4π c) -2πab

Problem 5: By means of line integral calculate area of the interior of

a) ellipse $\boldsymbol{r}(t) = a\cos t\,\boldsymbol{i} + b\sin t\,\boldsymbol{j}$, $t\in\langle 0,2\pi\rangle$

b) cycloid $\boldsymbol{r}(t) = a\,(t - \sin t)\,\boldsymbol{i} + a\,(1 - \cos t)\,\boldsymbol{j}$, $t\in\langle 0,2\pi\rangle$ bounded by axis x

c) asteroid $\boldsymbol{r}(t) = a\cos^3 t\,\boldsymbol{i} + a\sin^3 t\,\boldsymbol{j}$, $t\in\langle 0,2\pi\rangle$

d) Leaf of Descart $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}, t\in\langle 0,\infty\rangle$

Solution: a) $P = 4\pi ab$

b) $P = 3\pi a^2$

c) $P = 3\pi a^2/8$...

d) $P = 3a^2/2$