

Problems on function series

Problem 1: Find domain of convergence of the following power series:

a) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

b) $\sum_{n=1}^{\infty} n! x^n$

c) $\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{2^n}}$

d) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

e) $\sum_{n=1}^{\infty} \frac{x^n}{n + \sqrt{n}}$

f) $\sum_{n=1}^{\infty} \frac{10^n x^n}{\sqrt{n}}$

g) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

h) $\sum_{n=1}^{\infty} \frac{x^{n^2}}{2^n}$

i) $\sum_{n=1}^{\infty} x^n \left(1 + \frac{1}{n}\right)^{n^2}$

j) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n (n+1)(n+2)}$

Solutions: a) $(-\infty, \infty)$

b) $\{0\}$

c) $\left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$

d) $(-\infty, \infty)$

e) $\langle -1, 1 \rangle$

f) $\left\langle -\frac{1}{10}, \frac{1}{10} \right\rangle$

g) $\langle -1, 1 \rangle$

h) $(-2, 2)$

i) $\langle -1, 1 \rangle$

j) $\langle -3, 1 \rangle$

Problem 2: Find Taylor series of function f at the point a :

a) $f(x) = e^x, a = 1$

b) $f(x) = \ln(x + 3), a = -1$

c) $f(x) = \sin x, a = \pi/2$

d) $f(x) = \sqrt{x}, a = 4$

e) $f(x) = \frac{1}{x}, a = 2$

Solutions: a) $T(e^x, 1, x) = e \left(1 + \frac{x-1}{1!} + \frac{(x-1)^2}{2!} + \dots + \frac{(x-1)^n}{n!} + \dots \right)$

b) $T(\ln(x+3), -1, x) = \ln 2 + \frac{x+1}{2} - \frac{(x+1)^2}{8} + \dots + (-1)^{n+1} \frac{(x+1)^n}{n \cdot 2^n} + \dots$

c) $T(\sin x, \frac{\pi}{2}, x) = 1 - \frac{1}{2} \left(x - \frac{\pi}{2} \right)^2 + \frac{1}{24} \left(x - \frac{\pi}{2} \right)^4 + \dots + (-1)^n \frac{1}{(2n)!} \left(x - \frac{\pi}{2} \right)^{2n} + \dots$

d) $T(\sqrt{x}, 4, x) = 2 + \frac{1}{2 \cdot 1! \sqrt{4}} (x-4) - \frac{1}{2 \cdot 2! \sqrt{4^3}} (x-4)^2 + \dots + (-1)^{n+1} \frac{1 \cdot 3 \dots (2n-3)}{2^{2n-1} \cdot n! \sqrt{4^n}} (x-4)^n + \dots$

e) $T\left(\frac{1}{x}, 2, x\right) = \frac{1}{2} - \frac{1}{4} (x-2) + \frac{1}{4} (x-2)^2 + \dots + (-1)^n \frac{n!}{2^{n+1}} (x-2)^n + \dots$

Problem 3: By means of expansion of function f to the Taylor series evaluate approximate values:

a) $\sin\left(\frac{1}{2}\right)$

b) $\cos(10^\circ)$

c) $\arctan\left(\frac{1}{5}\right)$

d) $\ln\left(\frac{4}{3}\right)$

Solutions: a) $\sin\left(\frac{1}{2}\right) = \frac{23}{48}$

b) $\cos\left(\frac{\pi}{18}\right) = 0,984769$

c) $\arctan\left(\frac{1}{5}\right) = \frac{74}{375}$

d) $\ln\left(\frac{4}{3}\right) = \frac{5}{18}$