PROBLEMS ON FIELDS

Problem 1: Find domain of definition Ω of function f, calculate gradient of function f and its derivative in given direction $\boldsymbol{s}$ and determine their values a the point X_0

a) $f(x, y) = 2x^2 + 3y^2$, $\boldsymbol{s} = (0, 1)$, $X_0 = [1, 1]$
b) $f(x, y) = \dfrac{x+y}{1-xy}$, $\boldsymbol{s} = (-1, 1)$, $X_0 = [1, -1]$
c) $f(x, y) = \ln\sqrt{x^2+y^2}$, $\boldsymbol{s} = (3, 4)$, $X_0 = [-1, -1]$
d) $f(x, y, z) = 2x^3y^2z^4$, $\boldsymbol{s} = (1, 1, 1)$, $X_0 = [1, 1, 1]$
e) $f(x, y, z) = \sin(x + y^2 + z^3)$, $\boldsymbol{s} = (0, 0, -1)$, $X_0 = [0, 0, \pi]$
f) $f(x, y, z) = e^x y + \ln z$, $\boldsymbol{s} = (1, 1, -1)$, $X_0 = [0, 1, 1]$

Solution: a) $\Omega = \boldsymbol{R}^2$, $\boldsymbol{F}(X) = \operatorname{grad} f = (4x, 6y)$, $\boldsymbol{F}(X_0) = (4, 6)$, $f'_s(X) = 6y$, $f'_s(X_0) = 6$,

b) $\Omega = \{(x, y) \in \boldsymbol{R}^2, xy \neq 1\}$, $\boldsymbol{F}(X) = \operatorname{grad} f = \left(\dfrac{1+y^2}{(1-xy)^2}, \dfrac{1+x^2}{(1-xy)^2}\right)$, $\boldsymbol{F}(X_0) = (1/2, 1/2)$,

$f'_s(X) = \dfrac{x^2-y^2}{\sqrt{2}(1-xy)^2}$, $f'_s(X_0)$ not defined ,

c) $\Omega = \{(x, y) \in \boldsymbol{R}^2, x \vee y \neq 0\}$, $\boldsymbol{F}(X) = \operatorname{grad} f = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$, $\boldsymbol{F}(X_0) = (-1/2, -1/2)$,

$f'_s(X) = \dfrac{4y-3x}{5(x^2+y^2)}$, $f'_s(X_0) = -0.1$,

d) $\Omega = \boldsymbol{R}^3$, $\boldsymbol{F}(X) = \operatorname{grad} f = (6x^2y^2z^4, 4x^3yz^4, 8x^3y^2z^3)$, $\boldsymbol{F}(X_0) = (6, 4, 8)$,

$f'_s(X) = \dfrac{6x^2y^2z^4 + 4x^3yz^4 + 8x^3y^2z^3}{\sqrt{3}}$, $f'_s(X_0) = 6\sqrt{3}$,

e) $\Omega = \boldsymbol{R}3$, $\boldsymbol{F}(X) = \operatorname{grad} f = (\cos(x+y^2+z^3), 2y\cos(x+y^2+z^3), 3z^2\cos(x+y^2+z^3))$,

$\boldsymbol{F}(X_0) = (-1, -2\pi, -3\pi^2)$, $f'_s(X) = -3z^2\cos(x+y^2+z^3)$, $f'_s(X_0) = -3\pi^2\cos\pi^3$,

f) $\Omega = \{(x, y, z) \in \boldsymbol{R}^3, z > 0\}\boldsymbol{R}^3$, $\boldsymbol{F}(X) = \operatorname{grad} f = (e^x y, e^x, 1/z)$, $\boldsymbol{F}(X_0) = (1, 1, 1)$,

$f'_s(X) = \dfrac{ye^x + e^x - 1}{\sqrt{3}}$, $f'_s(X_0) = \dfrac{1}{\sqrt{3}}$.

Problem 2: Find all such points in the region Ω, at which gradient of the scalar field (Ω, f) defined by function f from Problem 1. vanishes on Ω, and all such points, at which the gradient reaches the maximal value.

Solution: a) point $[0, 0]$, $|grad\ f| = \sqrt{16x^2 + 36y^2}$, increases over any bounds, $\varnothing$,

b) $\varnothing$, $|grad\ f| = \dfrac{\sqrt{(1+x^2)^2 + (1+y^2)^2}}{(1-xy)^2}$, increases over any bounds, $\varnothing$

c) $\varnothing$, $|grad\ f| = \dfrac{1}{\sqrt{x^2+y^2}}$, has no stationary points, $\varnothing$

d) all points of coordinate planes $x = 0$, $y = 0$, $z = 0$,

$|grad\ f| = 2x^2yz^3\sqrt{9y^2z^2 + 4x^2z^2 + 16x^2y^2}$, increases over any bounds, $\varnothing$

e) points of equipotential surfaces, whose coordinates satisfy equations $x + y^2 + z^3 = k.\pi/2$, $k \in Z$, $|grad\ f| = \cos r\sqrt{1+4y^2+9z^4}$, $r = x + y^2 + z^3$, points, whose coordinates satisfy equations $x + y^2 + z^3 = 2k.\pi$, $k \in Z$

f) $\emptyset$, $|grad\ f| = \sqrt{e^{2x}(y^2+1)+\frac{1}{z^2}}$, no stationary points, $\emptyset$

Problem 3: Find divergence of a vector field defined by a differentiable vector function $\boldsymbol{F}$ on $\boldsymbol{R}^3$, evaluate value of divergence at the point X_0 and determine type of field (solenoidal, rotational)

a) $\boldsymbol{F}(x, y, z) = (x + y + z, y + z, z)$, $X_0 = [1, 1,1]$

b) $\boldsymbol{F}(x, y, z) = (x^2z, -2y^3z^2, xy^2z)$, $X_0 = [1, -1, 1]$

c) $\boldsymbol{F}(x, y, z) = (x^2y, 2xz, -2yz)$, $X_0 = [-1, -1, 0]$

d) $\boldsymbol{F}(x, y, z) = (y^2, -yz, 2x)$, $X_0 = [0, 0, 1]$

e) $\boldsymbol{F}(x, y, z) = (xyz, x + y + z, xy + yz)$, $X_0 = [0, 1, 1]$

f) $\boldsymbol{F}(x, y, z) = (x(1 + y + z), y(x + 1 + z), z(x + y + 1))$, $X_0 = [1, 1, 1]$

Solution: a) div $\boldsymbol{F} = 3$, rotational field

b) div $\boldsymbol{F} = 2xz - 6y^2z^2 + x\,y^2$, div $\boldsymbol{F}(1, -1, 1) = -3$, rotational field

c) div $\boldsymbol{F} = 2xy - 2y$, div $\boldsymbol{F}(-1, -1, 0) = 4$, rotational field with flow

d) div $\boldsymbol{F} = -z$, div $\boldsymbol{F}(0, 0, 1) = -1$, rotational field with flow

e) div $\boldsymbol{F} = yz + 1 + y$, div $\boldsymbol{F}(0, 1, 1) = 3$, rotational field with flow

f) div $\boldsymbol{F} = 3 + 2(x + y + z)$, div $\boldsymbol{F}(1, 1, 1) = 9$, rotational field with flow

Problem 4: Find all such points in $\boldsymbol{R}^3$, at which divergence of a vector field from Problem 3. vanishes.

Solution: a) div $\boldsymbol{F} = 3 \neq 0$, $\emptyset$

b) $2xz - 6y^2z^2 + x\,y^2 = 0$, quartic surface determined by the given equation

c) $y = 0$, coordinate plane xz and plane $x = 1$ parallel to coordinate plane yz

d) $z = 0$, coordinate plane xy

e) $y(z + 1) + 1 = 0$, quadric surface determined by the given equation

f) $2x + 2y + 2z + 3 = 0$, plane

Problem 5: Find curl of a vector field determined by a vector function $\boldsymbol{F}$ from Problem 3., evaluate its value at the point X_0 and determine whirls at the given point.

Solution: a) curl $\boldsymbol{F} = (-1, 1, -1)$, constant whirl at all points of the vector field with velocity $|\text{curl}\ \boldsymbol{F}| = v = \sqrt{3}$

b) curl $\boldsymbol{F} = (yz(2x + 4y^2), x^2 - y^2z, 0)$, curl $\boldsymbol{F}(1, -1,1) = (-6, 0, 0)$, whirl with velocity $v = 6$

c) curl $\boldsymbol{F} = (-2(x + z), 0, 2z - x^2)$, rot $\boldsymbol{F}(-1, -1,0) = (2, 0, -1)$, whirl with velocity $v = \sqrt{5}$

d) curl $\boldsymbol{F} = (y, -2, -2y)$, rot $\boldsymbol{F}(0, 0, 1) = (0, -2, 0)$, whirl with velocity $v = 2$

e) curl $\boldsymbol{F} = (x + z - 1, xy - y, xz + 1)$, rot $\boldsymbol{F}(0, 1, 1) = (0, -1, 1)$, whirl with velocity $v = \sqrt{2}$

f) curl $\boldsymbol{F} = (z - y, x - z, y - x)$, rot $\boldsymbol{F}(1, 1, 1) = (0, 0, 0)$, no whirl at the point

Problem 6: Find all such points in $\boldsymbol{R}^3$, at which curl of the vector field determined by the vector function $\boldsymbol{F}$ from Problem 3. vanishes (there exist no whirl at the point) and points, at which the rotation is maximal.

Solution: a) $\varnothing$, constant rotation $|\text{curl } \boldsymbol{F}| = \sqrt{3}$

b) $[0, 0, z]$ – points at axis z and $[0, y, 0]$ – points at axis y and points $[-1/2, \pm 1/2, 1]$, $|\text{curl } \boldsymbol{F}| = \sqrt{y^2z^2(2x+4y^2)^2+(x^2-y^2z)^2}$, increases over any bounds - $\varnothing$

c) $[0, y, 0]$ – points at axis y and $[-2, y, 2]$ – points on line parallel to axis y, $|\text{curl } \boldsymbol{F}| = \sqrt{4(x+z)^2+(2z-x^2)^2}$, increases over any bounds - $\varnothing$

d) $\varnothing$, $|\text{curl } \boldsymbol{F}| = \sqrt{4+5y^2}$, increases over any bounds - $\varnothing$

e) $\left[\frac{1-\sqrt{5}}{2}, \frac{2}{1-\sqrt{5}}, \frac{1+\sqrt{5}}{2}\right], \left[\frac{1+\sqrt{5}}{2}, \frac{2}{1+\sqrt{5}}, \frac{1-\sqrt{5}}{2}\right]$

$|\text{curl } \boldsymbol{F}| = \sqrt{(x+z-1)^2+(xy-1)^2+(xz+1)^2}$, increases over any bounds - $\varnothing$

f) $[1, 1, 1]$, $|\text{rot } \boldsymbol{F}| = \sqrt{(z-1)^2+(x-z)^2+(y-z)^2}$, increases over any bounds - $\varnothing$

Problem 7: Find out, whether the vector field determined by vector function $\boldsymbol{F}$ is irrotational and find points of vanishing divergence of function $\boldsymbol{F}$ and points, at which it is extremal:

a) $\boldsymbol{F}(x, y, z) = (2x^2 + 8xy^2z, 3x^3y - 3xy, -4y^2z^2 - 2x^3z)$

b) $\boldsymbol{F}(x, y, z) = (-2xz^3, -2yz^3, z^4)$

c) $\boldsymbol{F}(x, y, z) = (xy - x^3, x^3, -yz)$

d) $\boldsymbol{F}(x, y, z) = (x + y z^3, xy^2, -xyz^2)$

e) $\boldsymbol{F}(x, y, z) = (x^3, y^3, z^3)$

f) $\boldsymbol{F}(x, y, z) = (x^2 + y^2 + z^2, x^2y^2 + 3y^2z^2 + x^2z^2, -2xz - 2x^2yz - 2yz^3)$

Solution: a) div $\boldsymbol{F} = x^3 + x \neq 0$, not solenoidal, div $\boldsymbol{F} = 0$ at points $[0, y, z]$, at all other points it is increasing, $[\text{div } \boldsymbol{F}]' = 3x^2 + 1 > 0$

b) div $\boldsymbol{F} = -2z^3 - 2z^3 + 4z^3 = 0$, solenoidal field

c) div $\boldsymbol{F} = y - 3x^2 + 3x^2 - y = 0$, solenoidal field

d) div $\boldsymbol{F} = 1 + 2xy - 2xyz \neq 0$, not solenoidal field, div $\boldsymbol{F} = 0$ at points determined by equation $z = 1 + \frac{1}{2xy}$, no extremes

e) div $\boldsymbol{F} = 3x^2 + 3y^2 + 3z^2 \neq 0$, not solenoidal field, div $\boldsymbol{F} = 0$ at point $[0, 0, 0]$, at all other points it is increasing

f) div $\boldsymbol{F} = 2x + 2x^2y + 6yz^2 - 2x - 2x^2y - 6yz^2 = 0$, solenoidal field

Problem 8: Express Laplacian of the vector field determined by the vector function $\boldsymbol{F}$ form Problem 7. and find points, at which it is vanishing.

Solution: a) $\Delta\boldsymbol{F} = (4 + 16xz, 18xy, -12xz - 8z^2 - 8y^2)$, $\Delta\boldsymbol{F} \neq \boldsymbol{0}$ at all points of the field

b) $\Delta\boldsymbol{F}(-12xz, 12yz, 12z^2)$, $\Delta\boldsymbol{F} = \boldsymbol{0}$ at points $[x, y, 0]$ in the coordinate plane xy

c) $\Delta\boldsymbol{F} = (-6x, 6x, 0)$, $\Delta\boldsymbol{F} = \boldsymbol{0}$ at points $[0, y, z]$ of the coordinate plane yz

d) $\Delta\boldsymbol{F} = (6yz, 2x, -2xy)$, $\Delta\boldsymbol{F} = \boldsymbol{0}$ at points $[0, y, 0]$ of the coordinate axis y

e) $\Delta\boldsymbol{F} = (6x, 6y, 6z)$, $\Delta\boldsymbol{F} = \boldsymbol{0}$ at point $[0, 0, 0]$

f) $\Delta\boldsymbol{F} = (6, 4x^2 + 8y^2 + 8z^2, -16yz)$, $\Delta\boldsymbol{F} \neq \boldsymbol{0}$ at all points of the field