PROBLEMS ON SURFACES

Problem 1: For surfaces given by vector functions find - vector function of surface normals, the first and second fundamental forms of surface and their discriminants, and Gaussian curvature

a) $\boldsymbol{p}(u, \mathrm{v}) = (u, v, 0)$, $(u, \mathrm{v}) \in \boldsymbol{R}^2$ – plane
b) $\boldsymbol{p}(u, \mathrm{v}) = (r\cos u, r\sin u, v)$, $u \in \langle 0, 2\pi\rangle$, $v \in \boldsymbol{R}$, $r \in \boldsymbol{R}$ – rotational cylindrical surface
c) $\boldsymbol{p}(u, \mathrm{v}) = (rv\cos u, rv\sin u, v)$, $u \in \langle 0, 2\pi\rangle$, $v \in \boldsymbol{R}$, $r \in \boldsymbol{R}$ – rotational conical surface
d) $\boldsymbol{p}(u, \mathrm{v}) = (r\sin u \cos v, r\sin u \sin v, r\cos u)$, $u \in \langle 0, \pi\rangle$, $v \in \langle 0, 2\pi\rangle$, $r \in \boldsymbol{R}$ – sphere
e) $\boldsymbol{p}(u, \mathrm{v}) = (r\sin u \cos v, r\sin u \sin v, r\ln(\mathrm{tg}\, u/2) + r\cos u)$, $u \in (0, \pi)$, $v \in \langle 0, 2\pi\rangle$, $r \in \boldsymbol{R}$ – pseudosphere
f) $\boldsymbol{p}(u, \mathrm{v}) = (a\sin u \cos v, a\sin u \sin v, b\cos u)$, $u \in \langle 0, \pi\rangle$, $v \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – ellipsoid
g) $\boldsymbol{p}(u, \mathrm{v}) = (au \cos v, au \sin v, bu^2)$, $u \in \langle -1, 1\rangle$, $v \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – rotational paraboloid
h) $\boldsymbol{p}(u, \mathrm{v}) = (u, v, u.v)$, $(u, v) \in \boldsymbol{R}^2$ – hyperbolic paraboloid
i) $\boldsymbol{p}(u, \mathrm{v}) = (a\cosh u \cos v, a\cosh u \sin v, b\sinh u)$, $u \in \langle -a, a\rangle$, $v \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – rotational hyperboloid
j) $\boldsymbol{p}(u, \mathrm{v}) = (a\cosh u \cos v, a\cosh u \sin v, u)$, $u \in \boldsymbol{R}$, $v \in \langle 0, 2\pi\rangle$, $a \in \boldsymbol{R}$ – rotational catenoid
k) $\boldsymbol{p}(u, \mathrm{v}) = ((d + r\cos u) \cos v, (d + r\cos u) \sin v, r\sin u)$, $u, v \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – torus
l) $\boldsymbol{p}(u, \mathrm{v}) = (au \cos v, au \sin v, bv)$, $u \in \boldsymbol{R}$, $v \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – helicoid
m) $\boldsymbol{p}(u, \mathrm{v}) = (\cos u - v \sin u, \sin u + v \cos u, u + v)$, $u \in \langle 0, 2\pi\rangle$, $v \in \boldsymbol{R}$, v, – surface of tangents to helix

Solution: a) $\boldsymbol{n} = (1, 0, 0)$, $\varphi_1 = 1du^2 + 1dv^2$, $D_1 = 1$, $\varphi_2 = 0$, $D_2 = 0$, $K = 0$

b) $\boldsymbol{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2du^2 + 1dv^2$, $D_1 = r^2$, $\varphi_2 = -r^2du^2$, $D_2 = 0$, $K = 0$

c) $\boldsymbol{n} = \dfrac{1}{\sqrt{1+r^2}}(\cos u, \sin u, -r)$, $\varphi_1 = r^2v^2du^2 + r^2dv^2$, $D_1 = r^4v^2$,

$\varphi_2 = -\dfrac{v}{\sqrt{1+r^2}}du^2$, $D_2 = 0$, $K = 0$

d) $\boldsymbol{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2du^2 + r^2\sin^2 u\, dv^2$, $D_1 = r^4\sin^2 u$, $\varphi_2 = r \sin^2 u du^2 + r\, dv^2$, $D_2 = r^2\sin^2 u$, $K = 1/r^2$

e) $\boldsymbol{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2 \sin^2 u\, du^2 + r^2\mathrm{cotg}^2 u\, dv^2$, $D_1 = r^4\cos^2 u$, $\varphi_2 = r \cos^2 u\, du^2 - r\, dv^2$, $D_2 = -r^2\cos^2 u$, $K = -1/r^2$

f) $\boldsymbol{n} = \dfrac{1}{\sqrt{b^2\sin^2 u + a^2\cos^2 u}}(b\sin u\cos v, b\sin u\sin v, a\cos u)$,

$\varphi_1 = (a^2\cos^2 u + b^2\sin^2 u)du^2 + a^2\sin^2 u\, dv^2$, $D_1 = a^4\sin^2 u\,(a^2\cos^2 u + b^2\sin^2 u)$,

$\varphi_2 = \dfrac{-ab}{\sqrt{a^2\cos^2 u + b^2\sin^2 u}}du^2 + \dfrac{-ab\sin^2 u}{\sqrt{a^2\cos^2 u + b^2\sin^2 u}}dv^2$,

$$D_2 = \frac{a^2b^2\sin^2 u}{a^2\cos^2 u + b^2\sin^2 u}, K = \frac{b^2}{a^2(a^2\cos^2 u + b^2\sin^2 u)^2}$$

g) $\boldsymbol{n} = \dfrac{1}{\sqrt{4b^2u^2+a^2}}(-2bu\cos v, -2bu\sin v, a)$, $\varphi_1 = (a^2 + 4b^2u^2)du^2 + a^2u^2\, dv^2$,

$D_1 = a^2u^2(a^2 + 4b^2u^2)$, $\varphi_2 = \dfrac{2ab}{\sqrt{4b^2u^2+a^2}}du^2 + \dfrac{2abu^2}{\sqrt{4b^2u^2+a^2}}dv^2$,

$D_2 = \dfrac{4a^2b^2u^2}{4b^2u^2+a^2}$, $K = \dfrac{4b^2}{16b^4u^4-a^4}$

h) $\boldsymbol{n} = \dfrac{1}{\sqrt{1+u^2+v^2}}(-v, -u, 1)$, $\varphi_1 = (1+v^2)du^2 + uv\, du\, dv + (1+u^2)dv^2$, $D_1 = 1 + u^2 + v^2$,

$\varphi_2 = \dfrac{1}{\sqrt{1+u^2+v^2}}du\, dv$, $D_2 = \dfrac{-1}{1+u^2+v^2}$, $K = \dfrac{-1}{(1+u^2+v^2)^2}$

i) $\boldsymbol{n} = \dfrac{1}{\sqrt{a^2\sinh^2 u + b^2\cosh^2 u^2}}(-b\cos v, -b\sin v, a\sinh u)$,

$\varphi_1 = (a^2\sinh^2 u + b^2\cosh^2 u)du^2 + a^2\cosh^2 u\, dv^2$,

$D_1 = a^2\cosh^2 u(a^2\sinh^2 u + b^2\cosh^2 u)$,

$\varphi_2 = \dfrac{a(b\sinh^2 u - \cosh u)}{\sqrt{a^2\sinh^2 u + b^2\cosh^2 u}}du^2 + \dfrac{a\cosh u}{\sqrt{a^2\sinh^2 u + b^2\cosh^2 u}}dv^2$,

$D_2 = \dfrac{a^2\cosh u(b\sinh^2 u - \cosh u)}{a^2\sinh^2 u + b^2\cosh^2 u}$, $K = \dfrac{b\sinh^2 u - \cosh u}{\cosh u(a^2\sinh^2 u + b^2\cosh^2 u)}$

j) $\boldsymbol{n} = \dfrac{1}{\sqrt{1+r^2\sinh^2 v}}(\cos u, \sin u, -r\sinh v)$, $\varphi_1 = r^2\cosh^2 v\, du^2 + r^2\cosh^2 v\, dv^2$, $D_1 = r^4\cosh^4 v$,

$\varphi_2 = -\dfrac{1}{r\sqrt{1+r^2\sinh^2 v}}du^2 + \dfrac{1}{r\sqrt{1+r^2\sinh^2 v}}dv^2$, $D_2 = \dfrac{-1}{r^2(1+r^2\sinh^2 v)}$,

$K = \dfrac{-1}{r^6\cosh^4 v(1+r^2\sinh^2 v)}$

k) $\boldsymbol{n} = (-\cos u\cos v, -\cos u\sin v, -\sin u)$, $\varphi_1 = r^2du^2 + (d + r\cos u)^2dv^2$,

$D_1 = r^2(d + r\cos u)^2$, $\varphi_2 = \dfrac{1}{d + r\cos u}du^2 + \dfrac{\cos u}{r}dv^2$, $D_2 = \dfrac{\cos u}{r(d + r\cos u)}$,

$K = \dfrac{\cos u}{r(d + r\cos u)^3}$

l) $\boldsymbol{n} = \dfrac{1}{\sqrt{b^2+a^2u^2}}(b\sin v, b\cos v, au)$, $\varphi_1 = a^2du^2 + (b^2 + a^2u^2)\, dv^2$, $D_1 = a^2(b^2 + a^2u^2)$,

$\varphi_2 = \dfrac{-ab}{b^2+a^2u^2}du dv$, $D_2 = \dfrac{-a^2b^2}{(b^2+a^2u^2)^2}$, $K = \dfrac{-b^2}{(b^2+a^2u^2)^3}$

m) $\boldsymbol{n} = \dfrac{1}{\sqrt{2}}(-\sin u, \cos u, -1)$, $\varphi_1 = (2 + v^2)\, d\, u^2 + 2du\, dv + 2dv^2$, $D_1 = 2v^2$, $\varphi_2 = -1/2\, du^2$,

$D_2 = 0$, $K = 0$

Problem 2: Evaluate Cartesian coordinates of point T determined by given curvilinear coordinates on surfaces that are defined by vector functions from respective letter of problem 1., and find equation of tangent plane to surface at this point:

a) $T = \boldsymbol{p}(1, 1)$

b) $T = \boldsymbol{p}(\pi, \pi)$

c) $T = \boldsymbol{p}(\pi, 1)$

d) $T = \boldsymbol{p}(\pi/2, 0)$

e) $T = \boldsymbol{p}(\pi/2, \pi/2)$

f) $T = \boldsymbol{p}(\pi, \pi)$

g) $T = \boldsymbol{p}(1, 3\pi/2)$

h) $T = \boldsymbol{p}(2, 2)$

i) $T = \boldsymbol{p}(0, 0)$

j) $T = \boldsymbol{p}(0, \pi/2)$

k) $T = \boldsymbol{p}(\pi, \pi/2)$

l) $T = \boldsymbol{p}(1, \pi)$

m) $T = \boldsymbol{p}(\pi/2, 1)$

Solution: a) $T = [1, 1, 0]$, $z = 0$

b) $T = [-r, 0, \pi]$, $x/r + z - \pi + 1 = 0$

c) $T = [-r, 0, 1]$, $x + rz = 0$

d) $T = [r, 0, r]$, $y = 0$

e) $T = [0, r, 0]$, $y = r$

f) $T = [0, 0, -b]$, $z + b = 0$

g) $T = [0, a, b]$, $2by + az - 3ab = 0$

h) $T = [2, 2, 4]$, $2x + 2y - z - 4 = 0$

i) $T = [a, 0, 0]$, $x - a = 0$

j) $T = [0, a, 0]$, $x = 0$

k) $T = [0, d - r, 0]$, $y + r - d = 0$

l) $T = [-a, 0, b]$, $by + az - ab = 0$

m) $T = [-1, 1, 1+\pi/2]$, $2x + 2z - \pi = 0$

Problem 3: Calculate angle formed by iso-parametric curves on surfaces from problem 1.

Solution: a) $\cos\omega = \dfrac{F}{\sqrt{EG}} = \dfrac{0}{1} = 0, \omega = 90^\circ$

b) $\cos\omega = \dfrac{F}{\sqrt{EG}} = \dfrac{0}{r} = 0, \omega = 90^\circ$

c) $\cos\omega = \dfrac{F}{\sqrt{EG}} = \dfrac{0}{r^2 v} = 0, \omega = 90^\circ$

d) $\cos\omega = \dfrac{F}{\sqrt{EG}} = \dfrac{0}{r^2 \sin^2 u} = 0, \omega = 90^\circ$

e) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 \cos u} = 0, \omega = 90^\circ$

f) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{a^2 \sin u \sqrt{a^2 \cos^2 u + b^2 \sin^2 u}} = 0, \omega = 90^\circ$

g) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{au\sqrt{(a^2 + 4b^2u^2)}} = 0, \omega = 90^\circ$

h) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{uv}{\sqrt{(1+u^2)(1+v^2)}} = A, \omega = \arccos A$

i) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{a\cosh u(a^2 \cosh^2 u + b^2 \sinh^2 u)} = 0, \omega = 90^\circ$

j) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 \cosh^2 v} = 0, \omega = 90^\circ$

k) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{r4(d + r\cos u)^2} = 0, \omega = 90^\circ$

l) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{0}{a^2(b^2 + a^2u^2)} = 0, \omega = 90^\circ$

m) $\cos\omega = \frac{F}{\sqrt{EG}} = \frac{2}{\sqrt{2(2+v^2)}} = A, \omega = \arccos A$

Problem 4: Find area of surface patch from problem 1. defined on region $\Omega \subset \boldsymbol{R}^2$:

a) $\Omega = \langle -2, 2\rangle \times \langle 0, 3\rangle$

b) $\Omega = \langle 0, 2\pi\rangle \times \langle 0, 10\rangle$

c) $\Omega = \langle 0, 2\pi\rangle \times \langle 0, 1\rangle$

d) $\Omega = \langle 0, \pi\rangle \times \langle 0, 2\pi\rangle$

e) $\Omega = \langle -\pi/2, \pi/2\rangle \times \langle 0, 2\pi\rangle$

f) $\Omega = \langle 0, \pi\rangle \times \langle 0, 2\pi\rangle$

g) $\Omega = \langle 0, 1\rangle \times \langle 0, 2\pi\rangle$

h) $\Omega = \langle -1, 1\rangle^2$

i) $\Omega = \langle -1, 1\rangle \times \langle 0, 2\pi\rangle$

j) $\Omega = \langle 0, \ln(2)\rangle \times \langle 0, 2\pi\rangle$

k) $\Omega = \langle 0, 2\pi\rangle \times \langle 0, 2\pi\rangle$

l) $\Omega = \langle 0, 1\rangle \times \langle 0, 2\pi\rangle$

m) $\Omega = \langle 0, 1\rangle \times \langle 0, 2\pi\rangle$

Solution: a) $P = \iint_\Omega \sqrt{D_1}\, dudv = \int_0^3 \int_{-2}^2 dudv = 12$

b) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{10} \int\limits_0^{2\pi} r dudv = 20\pi r$

c) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{1} \int\limits_0^{2\pi} rv\sqrt{1+r^2}\, dudv = \pi r\sqrt{1+r^2}$

d) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{\pi} r^2 \sin u dudv = 4\pi r^2$

e) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos u dudv = 4\pi r^2$

f) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{\pi} a^2 \sin u \sqrt{a^2\cos^2 u + b^2 \sin^2 u}\, dudv = 2\pi(a^2 + \dfrac{ab\arccos\frac{a}{b}}{\sin(\arccos\frac{a}{b})})$

g) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{1} au\sqrt{a^2 + 4b^2u^2}\, dudv = \dfrac{\pi a}{6b^2}(\sqrt{(a^2+4b^2)^3} - a^2)$

h) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_{-1}^{1} \int\limits_{-1}^{1} \sqrt{1+u^2+v^2}\, dudv = \dfrac{-2\pi}{9} + \dfrac{4}{3}(\sqrt{3} + \ln(7+4\sqrt{3}))$

i)

$$P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_{-1}^{1} a\cosh u \sqrt{a^2\sinh^2 u + b^2\cosh^2 u}\, dudv =$$

$$= \frac{a\pi}{2e^2\sqrt{a^2+b^2}} \left((e^2-1)\sqrt{a^2+b^2}\sqrt{a^2(e^2-1)^2 + b^2(e^2+1)^2} + 4b^2e^2 \arctan\frac{\sqrt{a^2+b^2}(e^2-1)}{\sqrt{a^2(e^2-1)^2+b^2(e^2+1)^2}} \right)$$

j) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{1} r^2 \cosh^2 u dudv = \dfrac{3}{2}\pi r^2$

k) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{2\pi} r(d + r\cos u) dudv = 4\pi^2 rd$

l) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{1} a\sqrt{a^2u^2 + b^2}\, dudv = \pi\left(a\sqrt{a^2+4b^2} + 4b^2 \ln\dfrac{a+\sqrt{a^2+4b^2}}{2a} \right)$

m) $P = \iint\limits_{\Omega} \sqrt{D_1}\, dudv = \int\limits_0^{2\pi} \int\limits_0^{1} \sqrt{2} v dudv = \sqrt{2}\pi$