PROBLEMS ON CURVES

Problem 1: Space curve is determined by vector function

a) $\boldsymbol{r}(t) = (\sin t, \cos t, 2 + t^2)$
b) $\boldsymbol{r}(t) = (2t, -t^3, t^2 - 1)$
c) $\boldsymbol{r}(t) = (\cos t, 1/\cos t, \sin t)$
d) $r(t) = (\sin t, \cos t, \tan t)$
e) $\boldsymbol{r}(t) = (e^t, e^{2t}, t.e^t)$

Find elements of the Frenet – Serret moving reper (vectors ***t***, ***b***, ***n***) and trihedron (equations of lines - tangent, normal and binormal, equations of planes - osculating, rectification, normal) of the curve, derive functions of the first and the second curvature, and calculate coordinates of the curve point P with curvilinear coordinate $t = 0$, vectors ***t***, ***b***, ***n*** and values of curvatures ${}^1k$, ${}^2k$ in P.

Solution:

a) $P = [0,1,2], \boldsymbol{t}(0) = (1,0,0), \boldsymbol{b}(0) = (0,-\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}), \boldsymbol{n}(0) = (0,\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}), {}^1k(0) = \sqrt{5}, {}^2k(0) = 0$

b) $P = [0,0,-1], \boldsymbol{t}(0) = (1,0,0), \boldsymbol{b}(0) = (0,1,0), \boldsymbol{n}(0) = (0,0,-1), {}^1k(0) = 0.5, {}^2k(0) = 1.5$

c) $P = [1,1,0], \boldsymbol{t}(0) = (0,0,1), \boldsymbol{b}(0) = (-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0), \boldsymbol{n}(0) = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0), {}^1k(0) = \sqrt{2}, {}^2k(0) = 0$

d) $P = [0,1,0], \boldsymbol{t}(0) = (\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}), \boldsymbol{b}(0) = (\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}), \boldsymbol{n}(0) = (0,-1,0), {}^1k(0) = 0.5, {}^2k(0) = -1.5$

e) $P = [1,1,0], \boldsymbol{t}(0) = (\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}), \boldsymbol{b}(0) = (0,-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}), \boldsymbol{n}(0) = (-\frac{3}{\sqrt{30}},\frac{1}{\sqrt{30}},\frac{1}{\sqrt{30}}),$
${}^1k(0) = \frac{\sqrt{5}}{6\sqrt{6}}, {}^2k(0) = -2$

Problem 2: Determine function of the first curvature ${}^1k(t)$ of curve defined by vector function

a) $\boldsymbol{r}(t) = (a \cos t, b \sin t, 0)$, $t \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ - ellipse
b) $\boldsymbol{r}(t) = (a/\cos t, b \tan t, 0)$, $t \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ - hyperbola
c) $\boldsymbol{r}(t) = (t, pt^2, 0)$, $t \in \boldsymbol{R}$, $p \in \boldsymbol{R}$ - parabola
d) $\boldsymbol{r}(t) = (t, a^t, 0)$, $t \in \boldsymbol{R}$, $a \in \boldsymbol{R}$, $a \neq 1$ – exponential curve
e) $\boldsymbol{r}(t) = (t, \log_a t, 0)$, $t \in \boldsymbol{R}^+$, $a \in \boldsymbol{R}$, $a \neq 1$ - logarithmic curve
f) $\boldsymbol{r}(t) = (t, \sin t, 0)$, $t \in \boldsymbol{R}$ – sinusoidal curve
g) $\boldsymbol{r}(t) = (t, \tan t, 0)$, $t \in \langle -\pi/2, \pi/2\rangle$ - graph of function tangents
h) $\boldsymbol{r}(t) = (t, \sinh t, 0)$, $t \in \boldsymbol{R}$ – chain curve
i) $\boldsymbol{r}(t) = (a \cos t, b \sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – elliptic helix
j) $\boldsymbol{r}(t) = (r (1 - t) \cos 2\pi t, r (1 - t) \sin 2\pi, vt)$, $t \in \langle 0, 2\pi\rangle$, $r, v \in \boldsymbol{R}$ - conical helix
k) $\boldsymbol{r}(t) = (r \sin t \cos 2t, -r \sin t \sin 2t, r \cos t)$, $t \in \langle 0, \pi\rangle$, $r \in \boldsymbol{R}$ - spherical helix
l) $\boldsymbol{r}(t) = (-r \sin t \cos t, r \cos^2 t, r \sin t)$, $t \in \langle 0, 2\pi\rangle$, $r \in \boldsymbol{R}$ – Viviani curve
m) $\boldsymbol{r}(t) = (r \cos t, r \sin t, r \sin t \cos t)$, $t \in \langle 0, 2\pi \rangle$, $r \in \boldsymbol{R}$ - bended circle

and find curve points with the extremal curvature.

Solution: a) ${}^1k(t) = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$, major vertices $P(0)$, $P(\pi)$ - ${}^1k(t) = ab^{-1}$ - max, minor vertices $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t) = a^{-1}b$ - min

b) ${}^1k(t) = \frac{ab\cos^3 t}{(a^2 \sin^2 t + b^2)\sqrt{a^2 \sin^2 t + b^2}}$, major vertices $P(0)$, $P(\pi)$ - ${}^1k(t) = ab^{-2}$ - max

c) ${}^1k(t)=\dfrac{2p}{(1+4p^2t^2)\sqrt{1+4p^2t^2}}$, vertex $P(0)$ - ${}^1k(t)=2p$ - max

d) ${}^1k(t)=\dfrac{\ln a}{t^2(1+t^{-2}\ln^2 a)^{3/2}}$, no extrema

e) ${}^1k(t)=\dfrac{a^t\ln^2 a}{(1+\ln^2 a^{2t})^{3/2}}$, no extrema

f) ${}^1k(t)=\dfrac{\sin t}{(1+\cos^2 t)^{3/2}}$, $P(\pi/2)$ - ${}^1k(t)=1$ - max

g) ${}^1k(t)=\dfrac{2\sin t\cos^5 t}{(1+\cos^4 t)^{3/2}}$, $P(\pi/2)$ - ${}^1k(t)=1$ - max

h) ${}^1k(t)=\dfrac{\sinh t}{(1+\cosh^2 t)^{3/2}}$, $P(0)$ - ${}^1k(t)=1$ - max

i) ${}^1k(t)=\dfrac{\sqrt{a^2v^2\cos^2 t+b^2v^2\sin^2 t+a^2b^2}}{(a^2\sin^2 t+b^2\cos^2 t+v^2)^{3/2}}$,

$P(0)$, $P(\pi)$ - ${}^1k(t)=\dfrac{a}{b^2+v^2}$ - max, $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t)=\dfrac{b}{a^2+v^2}$ - min

j) ${}^1k(t)=\dfrac{rd_1}{d^3}, d=\sqrt{r^2(2-2t+t^2)+v^2}, d_1=\sqrt{r^2(3-2t+t^2)+v^2(5-2t+t^2)}$,

$P(1)$ - ${}^1k(t)=\dfrac{r\sqrt{2r^2+4v^2}}{(r^2+v^2)^{3/2}}$ - max, $P(t_1)$, $P(t_2)$ - ${}^1k(t)=\dfrac{r\sqrt{-r^2+\dfrac{rv^4}{r^2}-2v^2}}{\left(\dfrac{-3r^5+rv^4-6r^3v^2+2rv^4}{r(r^2+v^2)}\right)^{3/2}}$ –

min, $t_{1,2}=\dfrac{2r^5+2r^3v^2\pm\sqrt{2}\sqrt{-r^3(r^2+v^2)(5r^5-rv^4+10r^3v^2)}}{2r^5+2r^3v^2}$,

k) ${}^1k(t)=\dfrac{4\sqrt{4(-7+3\cos 2t)^2+(1-6\cos 2t+3\cos 4t)^2+576\cos^2 t\sin^4 t}}{r(\sqrt{(\cos t-3\cos^2)^2+4\sin^2 t+(\sin t-3\cos 3t)^2})^3}$,

$P(0)$, $P(\pi)$ - ${}^1k(t)=\dfrac{8}{13r}\sqrt{\dfrac{17}{13}}$ - max, $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t)=\dfrac{4}{5r}\sqrt{\dfrac{197}{5}}$ - max,

l) ${}^1k(t)=\dfrac{\sqrt{4(1+\cos^6 t)+\sin^2 t(2+\cos 2t)^2}}{r\sqrt{(1+\cos^2 t)^3}}$,

$P(0)$, $P(\pi)$ - ${}^1k(t)=\dfrac{1}{r}$ - max, $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t)=\dfrac{2\sqrt{2}}{r}$ - max

m) ${}^1k(t)=\dfrac{\sqrt{1+\cos^2 t(-2+\cos 2t)^2+(2+\sin t\cos 2t)^2}}{r\sqrt{(1+\cos^2 2t)^3}}$,

$P(0)$, $P(\pi)$ - ${}^1k(t)=\dfrac{1}{r}\sqrt{\dfrac{3}{4}}$ - min, $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t)=\dfrac{\sqrt{5}}{r}$ - max

Problem 3: Find all such values of parameter t that define points with the extremal second curvature on the curve, which is hodograph of the vector function

a) $\boldsymbol{r}(t) = (2t, t^2 - 1, 1 - t)$, $t \in \boldsymbol{R}$

b) $\boldsymbol{r}(t) = (\sin 2t, 2\sin t, \cos t)$

c) $\boldsymbol{r}(t) = (a \cos t, b \sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – elliptic helix

d) $\boldsymbol{r}(t) = (r (1 - t) \cos 2\pi t, r (1 - t) \sin 2\pi, vt)$, $t \in \langle 0, 1\rangle$, $r, v \in \boldsymbol{R}$ - conical helix

e) $\boldsymbol{r}(t) = (r \sin \pi t \cos 2\pi t, -r \sin \pi t \sin 2\pi t, r \cos \pi t)$, $t \in \langle 0, 1\rangle$, $r \in \boldsymbol{R}$ - spherical helix

f) $\boldsymbol{r}(t) = (-r \sin t \cos t, r \cos^2 t, r \sin t)$, $t \in \langle 0, 2\pi\rangle$, $r \in \boldsymbol{R}$ – Viviani curve

g) $\boldsymbol{r}(t) = (r \cos t, r \sin t, r \sin t \cos t)$, $t \in \langle 0, 2\pi \rangle$, $r \in \boldsymbol{R}$ – bended circle

Solution:.a) ${}^2k(t) = 0$, $t \in \boldsymbol{R}$

b) ${}^2k(t) = 0$, $t \in \boldsymbol{R}$

c) ${}^2k(t) = \dfrac{abv}{a^2v^2\cos^2 t + b^2v^2\sin^2 t + a^2b^2}$, $P(0)$, $P(\pi)$ - ${}^2k(t) = \dfrac{abv}{a^2v^2 + a^2b^2}$ - min,

$P(\pi/2)$, $P(3\pi/2)$ - ${}^2k(t) = \dfrac{abv}{b^2v^2 + a^2b^2}$ - max

d) ${}^2k(t) = \dfrac{7 - 2t + t^2}{r^2(3 - 2t + t^2) + v^2(5 - 2t + t^2)}$, $P(0)$ - ${}^2k(t) = \dfrac{7}{3r^2 + 5v^2}$ - min

e) ${}^2k(t) = \dfrac{-11\cos t + 3\cos 3t + 4(\sin 2t - 6\sin 4t + 3\sin 6t)}{r((-7 + 3\cos 2t)^2 + 4(1 - 6\cos 2t + 3\cos 4t)^2 + 576\sin^6 t\cos^2 t))}$,

$P(0)$ - max, $P(\pi)$ – min, ${}^2k(t) = \dfrac{\pm 4}{17r}$

f) ${}^2k(t) = \dfrac{6\cos t}{r(4(1 + \cos^6 t) + \sin^2 t(2 + \cos 2t)^2)}$, $P(0)$ - max, $P(\pi)$ – max, ${}^2k(t) = \dfrac{\pm 3}{4r}$

g) ${}^2k(t) = \dfrac{12\cos 2t}{r(4(-7 + 3\cos 2t)^2 + (1 - 6\cos 2t + 3\cos 4t)^2 + 576\cos^2 t\sin^6 t)}$

$P(0)$ - max, $P(\pi)$ – max, ${}^2k(t) = \dfrac{12}{17r}$, $P(\pi/2)$, $P(3\pi/2)$ –min, ${}^2k(t) = 0$

Problem 4: Find out which of the curves determined by vector functions

a) $\boldsymbol{r}(t) = (r \cos t, r \sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $r \in \boldsymbol{R}$ – cylindrical helix

b) $\boldsymbol{r}(t) = (a \cos t, b \sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $a, b \in \boldsymbol{R}$ – elliptical helix

c) $\boldsymbol{r}(t) = (r (1 - t) \cos t, r (1 - t) \sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $r, v \in \boldsymbol{R}$ – conical helix

d) $\boldsymbol{r}(t) = (r \sin t \cos 2t, -r \sin t \sin 2t, r \cos t)$, $t \in \langle 0, \pi\rangle$, $r \in \boldsymbol{R}$ – spherical helix

are curves of constant slope.

Solution: a) $\dfrac{{}^1k(t)}{{}^2k(t)} = \dfrac{r}{v}$

b) $\dfrac{{}^1k(t)}{{}^2k(t)} = \dfrac{1}{abv}\left(\dfrac{a^2v^2\cos^2 t + b^2v^2\sin^2 t + a^2b^2}{a^2\sin^2 t + b^2\cos^2 t + v^2}\right)^{\frac{3}{2}}$, not a constant slope curve

c) $\dfrac{{}^1k(t)}{{}^2k(t)} = \dfrac{d_1^3}{rd^3} = \dfrac{1}{r}\left(\dfrac{d_1}{d}\right)^3$

d) $\frac{{}^1k(t)}{{}^2k(t)} = \frac{1}{r}\left(\frac{d_2}{d_3}\right)^3$

Problem 5: Find vector function of curve that is the equidistant of curve determined by vector function

a) $\boldsymbol{r}(t) = (t, at^2 + bt + c, 0)$, $t \in \boldsymbol{R}$, $a, b, c \in \boldsymbol{R}$
b) $\boldsymbol{r}(t) = (t, \cos t, 0)$, $t \in \langle 0, 2\pi\rangle$
c) $\boldsymbol{r}(t) = (t, \ln t, 0)$, $t \in (0, \infty\rangle$
d) $\boldsymbol{r}(t) = (r\cos t, r\sin t, vt)$, $t \in \langle 0, 2\pi\rangle$, $r, v \in \boldsymbol{R}$
e) $\boldsymbol{r}(t) = ((1 - t)\cos t, 0, vt)$, $t \in \langle 0, 2\pi\rangle$, $v \in \boldsymbol{R}$

Solution: a) $\boldsymbol{p}(t) = \left(t \pm k.\frac{2at}{\sqrt{1+(2at+b)^2}}, at^2 + bt \pm k.\frac{1}{\sqrt{1+(2at+b)^2}}, 0\right), k \in \boldsymbol{R}$

b) $\boldsymbol{p}(t) = \left(t \pm k.\frac{\sin t}{\sqrt{1+\sin^2 t}}, \cos t \pm k.\frac{-1}{\sqrt{1+\sin^2 t}}, 0\right), k \in \boldsymbol{R}$

c) $\boldsymbol{p}(t) = \left(t \pm k.\frac{1}{\sqrt{1+t^2}}, \ln t \pm k.\frac{-t}{\sqrt{1+t^2}}, 0\right), k \in \boldsymbol{R}$

d) $\boldsymbol{p}(t) = (\cos t(r \pm k.\sqrt{r^2+v^2}, \sin t(r \pm k.\sqrt{r^2+v^2}, v.t), k \in \boldsymbol{R}$

e) $\boldsymbol{p}(t) = \left((1-t)\cos t \pm \frac{vk}{bd}, 0, v.t \pm \frac{k(\cos t + (1-t)\sin t)}{bd}\right), k \in \boldsymbol{R}$

$d = \sqrt{(\cos t + (1-t)\sin t)^2 + v^2}, b = 2\sin t + (t-1)\cos t$

Problem 6: Find vector function of quadratic curve defined on interval $\langle 0, 1\rangle$, with the start point P(0) and end point P(1), while vector $\boldsymbol{d}(t)$ is direction vector of tangent to curve in the point $P(t)$

a) $P(0) = [2, 2, 0]$, $P(1) = [0, 2, 2]$, $\boldsymbol{d}(0) = (2, 3, -2)$
b) $P(0) = [4, 0, 0]$, $P(1) = [0, 4, 0]$, $\boldsymbol{d}(1) = (2, 1, 3)$

Solution: a) $\boldsymbol{r}(t) = (-4t^2 + 2t + 2, -3t^2 + 3t + 2, 4t^2 - 2t)$

b) $\boldsymbol{r}(t) = (6t^2 - 10t + 4, -3t^2 + 7t, -3t^2 - 3t)$

Problem 7: Find vector function of cubic curve defined on interval $\langle 0, 1\rangle$, with the start point P(0) and end point P(1), while vectors $\boldsymbol{d}(0)$ and $\boldsymbol{d}(1)$ are direction vectors of tangent lines to curve in the respective points

a) $P(0) = [2, 1, 0]$, $P(1) = [1, 4, 2]$, $\boldsymbol{d}(0) = (0, 0, 2)$, $\boldsymbol{d}(1) = (0, 2, 0)$
b) $P(0) = [0, 0, 4]$, $P(1) = [0, 0, 0]$, $\boldsymbol{d}(0) = (1, 1, -1)$, $\boldsymbol{d}(1) = (-1, -1, 0)$

Solution: a) $\boldsymbol{r}(t) = (2t^3 - 3t^2 + 2, -4t^3 + 7t^2 + 1, -2t^3 + 2t^2 + 2t)$
b) $\boldsymbol{r}(t) = (-t^2 + t, -2t^3 + t^2 + t, 8t^3 - 12t^2 + 4)$

Problem 8: Find vector function of cubic curve defined on interval $\langle 0, 1\rangle$, with the start point P(0) and end point P(1), while vector $\boldsymbol{d}(0)$ is direction vector of tangent line and vector $\boldsymbol{b}(0)$ is direction vector of binormal to the curve in the point P(0)

a) $P(0) = [1, 0, 0]$, $P(1) = [0, 1, 1]$, $\boldsymbol{d}(0) = (1, 0, 0)$, $\boldsymbol{b}(0) = (0, 0, -1)$
b) $P(0) = [2, 2, 0]$, $P(1) = [0, 0, 2]$, $\boldsymbol{d}(0) = (0, 1, 0)$, $\boldsymbol{b}(0) = (0, 0, 1)$

Solution: a) $\boldsymbol{r}(t) = (at^3 + (-2 - a)t^2 + t + 1, 1\frac{1}{2}t^3 - \frac{1}{2}t^2, t^3)$
b) $\boldsymbol{r}(t) = (-1\frac{1}{2}t^3 - \frac{1}{2}t^2 + 2, at^3 - (a + 3)t^2 + t + 2, 2t^3)$