PROBLEMS ON VECTOR FUNCTIONS II

Problem 1: Calculate limits

a) $\lim\limits_{t\to 0}\left[\left(\frac{\sin t}{t},\frac{\cos t-1}{2t},e^{t}\right)\right]$ b) $\lim\limits_{t\to 0}\left[\left(\frac{1-\sqrt{t}}{1-t},\frac{t}{1+t},1\right)\right]$

c) $\lim\limits_{t\to \pi}\left[\left(\frac{\sin t}{t-\pi},\frac{1+\cos t}{t},\frac{t}{\pi}\right)\right]$ d) $\lim\limits_{t\to 1}\left[\left(\frac{e^{t}-e}{t-1},\frac{\ln t-1}{1-t},2t\right)\right]$

Solution: a) (1,0,1), b) (1,0,1), c) (−1,0,1), d) (e, −1,2).

Problem 2: Let $\omega \in R$ be arbitrary real number and **a**, **b** be arbitrary vectors, and let a vector function be determined by formula $\mathbf{r}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$. Prove that the following equation holds $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}+\omega^2\mathbf{r}=\mathbf{0}$.

Solution:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}=[-\mathbf{a}\omega\sin\omega t+\mathbf{b}\omega\cos\omega t]'=-\mathbf{a}\omega^2\cos\omega t-\mathbf{b}\omega^2\sin\omega t=-\omega^2(\mathbf{a}\cos\omega t+\mathbf{b}\sin\omega t)=-\omega^2\mathbf{r}.$$

Problem 3: Let $\omega \in R$ be arbitrary real number and **a**, **b** be arbitrary vectors, and let a vector function be determined by formula $\mathbf{r}(t) = \mathbf{a}\, e^{\omega t} + \mathbf{b}\, e^{-\omega t}$. Prove that the following equation holds $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}-\omega^2\mathbf{r}=\mathbf{0}$.

Solution: $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}=[\mathbf{a}\omega e^{\omega t}-\mathbf{b}\omega e^{-\omega t}]'=\mathbf{a}\omega^2e^{\omega t}+\mathbf{b}\omega^2e^{-\omega t}=\omega^2(\mathbf{a}e^{\omega t}+\mathbf{b}e^{-\omega t})=\omega^2\mathbf{r}$.

Problem 4: Determine size o velocity and accelerationof a point moving on trajectory determined by vector function $\mathbf{r}(t) = (e^t, 2\cos 3t, 2\sin 3t)$ in time $t = 0$.

Solution: $|\boldsymbol{v}(0)|=\sqrt{37}, |\boldsymbol{a}(0)|=\sqrt{325}$.

Problem 5: Determine size o velocity and accelerationof a point moving on trajectory determined by vector function $\boldsymbol{r}(t) = (e^{-t}, \ln(t^2+1), -\tan t)$ in time $t = 0$.

Solution: $\mathbf{v}(0)=(-1,0,-1), \mathbf{a}(0)=(1,2,0)$.

Problem 6: Determine trajectory $\mathbf{r}(t)$ of a moving point, if its acceleration is given by vector function $\mathbf{a}(t) = (6t, -24t^2, 4\sin t)$, and its position and velocity are defined in the time $t = 0$, $\mathbf{r}(0) = (2,1,0)$, $\mathbf{v}(0) = (-1,0,3)$.

Solution: $\mathbf{r}(t)=(t^3-t+2,1-2t^4,7t-4\sin t)$.

Problem 7: Find antiderivatives of vector functions:

a) $\mathbf{r}(t)=(te^{t},\sin^2 t,-\frac{1}{1+t^2})$

b) $\mathbf{r}(t)=(\frac{1}{2}t^2,-t\sin t,2^{t})$

c) $\mathbf{r}(t)=(\frac{t}{1+t^2},te^{t^2},\cos t)$

d) $\mathbf{r}(t)=(\frac{1}{\cos^2 t},\frac{3^t}{\ln 3},2\sin t\cos t)$

Solution: a) $\mathbf{R}(t)=\left((t-1)e^t+c_1,\frac{1}{2}(t-\frac{1}{2}\sin t)+c_2,-\arctan t+c_3\right)$,

b) $\mathbf{R}(t)=\left(\frac{t^3}{6}+c_1,(t\cos t-\sin t)+c_2,\frac{2^t}{\ln 2}+c_3\right)$,

c) $\mathbf{R}(t)=\left(\frac{1}{2}\ln(1+t^2)+c_1,\frac{1}{2}e^{t^2}+c_2,\sin t+c_3\right)$,

d) $\mathbf{R}(t)=(\tan t+c_1,3^t+c_2,\sin^2 t+c_3)$.

Problem 8: Evaluate $\int_\alpha^\beta \mathbf{r}(t)\mathrm{d}t$, for vector functions

a) $\mathbf{r}(t) = \sin^2 t\cos t\,\mathbf{i} + \cos^2 t\sin t\,\mathbf{j} + \mathbf{k}$, $\alpha = 0$, $\beta = \pi$,
b) $\mathbf{r}(t) = (2t + \pi)\,\mathbf{i} + t\sin t\,\mathbf{j} + \pi\,\mathbf{k}$, $\alpha = 0$, $\beta = \pi$,
c) $\mathbf{r}(t)=\left(\frac{1}{2}e^{-\frac{t}{2}},\frac{1}{2}e^{\frac{t}{2}},e^t\right),\alpha=0,\beta=1$
d) $\mathbf{r}(t) = 4\sin t\cos t\,\mathbf{i} + (3t^2 - 2t + 1)\,\mathbf{j} + t^{-1}\mathbf{k}$, $\alpha = 0$, $\beta = 2\pi$.

Solution: a) $\frac{2}{3}\mathbf{j}+\pi\,\mathbf{k}$, b) $2\pi^2\mathbf{i}+\pi\,\mathbf{j}+\pi^2\mathbf{k}$,

c) $(1\text{-}e^{\frac{1}{2}})\mathbf{i}+(e^{\frac{1}{2}}-1)\mathbf{j}+(e-1)\mathbf{k}$, d) $(0{,}7\pi - \pi^2 + 3\pi, \ln 2)$.

Problem 9: By means of Taylor polynomial of order 4 approximate value of vector function $\mathbf{r}(t)=((t-1)e^t,t+\sin t,\ln t)$ at the point $t = 0$.

Solution:

$$\mathbf{r}(t)=\mathbf{r}(0)+\frac{\mathbf{r}'(0)}{1!}(t-0)+\frac{\mathbf{r}''(0)}{2!}(t-0)^2+\frac{\mathbf{r}'''(0)}{3!}(t-0)^3+\frac{\mathbf{r}^{(4)}(0)}{4!}(t-0)^4=$$

$$=(-1{,}0{,}0)+(0{,}2{,}1)t+\frac{1}{2}(1{,}0{,}-1)t^2+\frac{1}{6}(2{,}-1{,}2)t^3+\frac{1}{24}(3{,}0{,}-6)t^4=$$

$$=(-1+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{1}{8}t^4,2t-\frac{1}{6}t^3,t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4)$$

Problem 10: Find value of variable $\xi \in \langle 2, 3\rangle$, at which the mean value theorem holds for values of vector function $\mathbf{r}(t) = (t^2 - 2t + 1, t^2 - 1)$ on given interval, therefore the inequality holds $|\mathbf{r}(\beta)-\mathbf{r}(\alpha)|\le(\beta-\alpha)|\mathbf{r}'(\xi)|$.

Solution: $\xi = 5/2$.