

ÚLOHY O PLOCHÁCH

Príklad 1: Pre uvedené plochy nájdite - vektorovú funkciu normál plochy, prvú a druhú základnú formu plochy a ich diskriminanty a Gaussovu krivosť plochy

- $\mathbf{p}(u, v) = (u, v, 0)$, $(u, v) \in \mathbf{R}^2$ - rovina
- $\mathbf{p}(u, v) = (r \cos u, r \sin u, v)$, $u \in \langle 0, 2\pi \rangle$, $v \in \mathbf{R}$, $r \in \mathbf{R}$ - rotačná valcová plocha
- $\mathbf{p}(u, v) = (rv \cos u, rv \sin u, v)$, $u \in \langle 0, 2\pi \rangle$, $v \in \mathbf{R}$, $r \in \mathbf{R}$ - rotačná kužeľová plocha
- $\mathbf{p}(u, v) = (r \sin u \cos v, r \sin u \sin v, r \cos u)$, $u \in \langle 0, \pi \rangle$, $v \in \langle 0, 2\pi \rangle$, $r \in \mathbf{R}$ - guľová plocha
- $\mathbf{p}(u, v) = (r \sin u \cos v, r \sin u \sin v, r \ln(\operatorname{tg} u/2) + r \cos u)$, $u \in (0, \pi)$, $v \in \langle 0, 2\pi \rangle$, $r \in \mathbf{R}$ - pseudosféra
- $\mathbf{p}(u, v) = (a \sin u \cos v, a \sin u \sin v, b \cos u)$, $u \in \langle 0, \pi \rangle$, $v \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - rotačný elipsoid
- $\mathbf{p}(u, v) = (au \cos v, au \sin v, bu^2)$, $u \in \langle -1, 1 \rangle$, $v \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - rotačný paraboloid
- $\mathbf{p}(u, v) = (u, v, u \cdot v)$, $(u, v) \in \mathbf{R}^2$ - hyperbolický paraboloid
- $\mathbf{p}(u, v) = (a \cosh u \cos v, a \cosh u \sin v, b \sinh u)$, $u \in \langle -a, a \rangle$, $v \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - rotačný hyperboloid
- $\mathbf{p}(u, v) = (a \cosh u \cos v, a \cosh u \sin v, u)$, $u \in \mathbf{R}$, $v \in \langle 0, 2\pi \rangle$, $a \in \mathbf{R}$ - rotačný katenoid
- $\mathbf{p}(u, v) = ((d + r \cos u) \cos v, (d + r \cos u) \sin v, r \sin u)$, $u, v \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - rotačný anuloid
- $\mathbf{p}(u, v) = (au \cos v, au \sin v, bv)$, $u \in \mathbf{R}$, $v \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - priamková skrútková plocha
- $\mathbf{p}(u, v) = (\cos u - v \sin u, \sin u + v \cos u, u + v)$, $u \in \langle 0, 2\pi \rangle$, $v \in \mathbf{R}$, v - plocha dotýčnic skrútkovice

Riešenie: a) $\mathbf{n} = (1, 0, 0)$, $\varphi_1 = 1 du^2 + 1 dv^2$, $D_1 = 1$, $\varphi_2 = 0$, $D_2 = 0$, $K = 0$

b) $\mathbf{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2 du^2 + 1 dv^2$, $D_1 = r^2$, $\varphi_2 = -r du^2$, $D_2 = 0$, $K = 0$

c) $\mathbf{n} = \frac{1}{\sqrt{1+r^2}}(\cos u, \sin u, -r)$, $\varphi_1 = r^2 v^2 du^2 + r^2 dv^2$, $D_1 = r^4 v^2$,

$\varphi_2 = -\frac{v}{\sqrt{1+r^2}} du^2$, $D_2 = 0$, $K = 0$

d) $\mathbf{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2 du^2 + r^2 \sin^2 u dv^2$, $D_1 = r^4 \sin^2 u$, $\varphi_2 = r \sin^2 u du^2 + r dv^2$,
 $D_2 = r^2 \sin^2 u$, $K = 1/r^2$

e) $\mathbf{n} = (\cos u, \sin u, 0)$, $\varphi_1 = r^2 \sin^2 u du^2 + r^2 \cot^2 u dv^2$, $D_1 = r^4 \cos^2 u$,
 $\varphi_2 = r \cos^2 u du^2 - r dv^2$, $D_2 = -r^2 \cos^2 u$, $K = -1/r^2$

f) $\mathbf{n} = \frac{1}{\sqrt{b^2 \sin^2 u + a^2 \cos^2 u}}(b \sin u \cos v, b \sin u \sin v, a \cos u)$,

$\varphi_1 = (a^2 \cos^2 u + b^2 \sin^2 u) du^2 + a^2 \sin^2 u dv^2$, $D_1 = a^4 \sin^2 u (a^2 \cos^2 u + b^2 \sin^2 u)$,

$\varphi_2 = \frac{-ab}{\sqrt{a^2 \cos^2 u + b^2 \sin^2 u}} du^2 + \frac{-ab \sin^2 u}{\sqrt{a^2 \cos^2 u + b^2 \sin^2 u}} dv^2$,

$D_2 = \frac{a^2 b^2 \sin^2 u}{a^2 \cos^2 u + b^2 \sin^2 u}$, $K = \frac{b^2}{a^2 (a^2 \cos^2 u + b^2 \sin^2 u)^2}$

$$g) \mathbf{n} = \frac{1}{\sqrt{4b^2u^2 + a^2}}(-2bu \cos v, -2bu \sin v, a), \varphi_1 = (a^2 + 4b^2u^2)du^2 + a^2u^2 dv^2,$$

$$D_1 = a^2u^2(a^2 + 4b^2u^2), \varphi_2 = \frac{2ab}{\sqrt{4b^2u^2 + a^2}}du^2 + \frac{2abu^2}{\sqrt{4b^2u^2 + a^2}}dv^2,$$

$$D_2 = \frac{4a^2b^2u^2}{4b^2u^2 + a^2}, K = \frac{4b^2}{16b^4u^4 - a^4}$$

$$h) \mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}}(-v, -u, 1), \varphi_1 = (1+v^2)du^2 + uv du dv + (1+u^2)dv^2, D_1 = 1 + u^2 + v^2,$$

$$\varphi_2 = \frac{1}{\sqrt{1+u^2+v^2}}du dv, D_2 = \frac{-1}{1+u^2+v^2}, K = \frac{-1}{(1+u^2+v^2)^2}$$

$$i) \mathbf{n} = \frac{1}{\sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u}}(-b \cos v, -b \sin v, a \sinh u),$$

$$\varphi_1 = (a^2 \sinh^2 u + b^2 \cosh^2 u)du^2 + a^2 \cosh^2 u dv^2,$$

$$D_1 = a^2 \cosh^2 u (a^2 \sinh^2 u + b^2 \cosh^2 u),$$

$$\varphi_2 = \frac{a(b \sinh^2 u - \cosh u)}{\sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u}}du^2 + \frac{a \cosh u}{\sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u}}dv^2,$$

$$D_2 = \frac{a^2 \cosh u (b \sinh^2 u - \cosh u)}{a^2 \sinh^2 u + b^2 \cosh^2 u}, K = \frac{b \sinh^2 u - \cosh u}{\cosh u (a^2 \sinh^2 u + b^2 \cosh^2 u)}$$

$$j) \mathbf{n} = \frac{1}{\sqrt{1+r^2 \sinh^2 v}}(\cos u, \sin u, -r \sinh v), \varphi_1 = r^2 \cosh^2 v du^2 + r^2 \cosh^2 v dv^2, D_1 = r^4 \cosh^4 v,$$

$$\varphi_2 = -\frac{1}{r\sqrt{1+r^2 \sinh^2 v}}du^2 + \frac{1}{r\sqrt{1+r^2 \sinh^2 v}}dv^2, D_2 = \frac{-1}{r^2(1+r^2 \sinh^2 v)},$$

$$K = \frac{-1}{r^6 \cosh^4 v (1+r^2 \sinh^2 v)}$$

$$k) \mathbf{n} = (-\cos u \cos v, -\cos u \sin v, -\sin u), \varphi_1 = r^2 du^2 + (d + r \cos u)^2 dv^2,$$

$$D_1 = r^2(d + r \cos u)^2, \varphi_2 = \frac{1}{d + r \cos u}du^2 + \frac{\cos u}{r}dv^2, D_2 = \frac{\cos u}{r(d + r \cos u)},$$

$$K = \frac{\cos u}{r(d + r \cos u)^3}$$

$$l) \mathbf{n} = \frac{1}{\sqrt{b^2 + a^2 u^2}}(b \sin v, b \cos v, au), \varphi_1 = a^2 du^2 + (b^2 + a^2 u^2) dv^2, D_1 = a^2(b^2 + a^2 u^2),$$

$$\varphi_2 = \frac{-ab}{b^2 + a^2 u^2}dudv, D_2 = \frac{-a^2 b^2}{(b^2 + a^2 u^2)^2}, K = \frac{-b^2}{(b^2 + a^2 u^2)^3}$$

$$m) \mathbf{n} = \frac{1}{\sqrt{2}}(-\sin u, \cos u, -1), \varphi_1 = (2 + v^2)du^2 + 2du dv + 2dv^2, D_1 = 2v^2, \varphi_2 = -1/2 du^2,$$

$$D_2 = 0, K = 0$$

Príklad 2: Nájdite karteziánske súradnice bodu T určeného krivočiarymi súradnicami na ploche danej vektorovou funkciou z príkladu 1. a rovnicu dotykovej roviny plochy v tomto bode:

a) $T = \mathbf{p}(1, 1)$

b) $T = \mathbf{p}(\pi, \pi)$

- c) $T = \mathbf{p}(\pi, 1)$
- d) $T = \mathbf{p}(\pi/2, 0)$
- e) $T = \mathbf{p}(\pi/2, \pi/2)$
- f) $T = \mathbf{p}(\pi, \pi)$
- g) $T = \mathbf{p}(1, 3\pi/2)$
- h) $T = \mathbf{p}(2, 2)$
- i) $T = \mathbf{p}(0, 0)$
- j) $T = \mathbf{p}(0, \pi/2)$
- k) $T = \mathbf{p}(\pi, \pi/2)$
- l) $T = \mathbf{p}(1, \pi)$
- m) $T = \mathbf{p}(\pi/2, 1)$

Riešenie: a) $T = [1, 1, 0], z = 0$

- b) $T = [-r, 0, \pi], x/r + z - \pi + 1 = 0$
- c) $T = [-r, 0, 1], x + rz = 0$
- d) $T = [r, 0, r], y = 0$
- e) $T = [0, r, 0], y = r$
- f) $T = [0, 0, -b], z + b = 0$
- g) $T = [0, a, b], 2by + az - 3ab = 0$
- h) $T = [2, 2, 4], 2x + 2y - z - 4 = 0$
- i) $T = [a, 0, 0], x - a = 0$
- j) $T = [0, a, 0], x = 0$
- k) $T = [0, d - r, 0], y + r - d = 0$
- l) $T = [-a, 0, b], by + az - ab = 0$
- m) $T = [-1, 1, 1 + \pi/2], 2x + 2z - \pi = 0$

Príklad 3: Nájdite uhol parametrických čiar plôch z príkladu 1.

Riešenie: a) $\cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{1} = 0, \omega = 90^\circ$

b) $\cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r} = 0, \omega = 90^\circ$

c) $\cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 v} = 0, \omega = 90^\circ$

d) $\cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 \sin^2 u} = 0, \omega = 90^\circ$

$$e) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 \cos u} = 0, \omega = 90^\circ$$

$$f) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{a^2 \sin u \sqrt{a^2 \cos^2 u + b^2 \sin^2 u}} = 0, \omega = 90^\circ$$

$$g) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{au \sqrt{(a^2 + 4b^2 u^2)}} = 0, \omega = 90^\circ$$

$$h) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{uv}{\sqrt{(1+u^2)(1+v^2)}} = A, \omega = \arccos A$$

$$i) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{a \cosh u (a^2 \cosh^2 u + b^2 \sinh^2 u)} = 0, \omega = 90^\circ$$

$$j) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^2 \cosh^2 v} = 0, \omega = 90^\circ$$

$$k) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{r^4 (d + r \cos u)^2} = 0, \omega = 90^\circ$$

$$l) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{0}{a^2 (b^2 + a^2 u^2)} = 0, \omega = 90^\circ$$

$$m) \cos \omega = \frac{F}{\sqrt{EG}} = \frac{2}{\sqrt{2(2+v^2)}} = A, \omega = \arccos A$$

Príklad 4: Nájdite obsah listu plôch z príkladu 1. definovaného na danej oblasti $\Omega \subset \mathbf{R}^2$:

$$a) \Omega = \langle -2, 2 \rangle \times \langle 0, 3 \rangle$$

$$b) \Omega = \langle 0, 2\pi \rangle \times \langle 0, 10 \rangle$$

$$c) \Omega = \langle 0, 2\pi \rangle \times \langle 0, 1 \rangle$$

$$d) \Omega = \langle 0, \pi \rangle \times \langle 0, 2\pi \rangle$$

$$e) \Omega = \langle -\pi/2, \pi/2 \rangle \times \langle 0, 2\pi \rangle$$

$$f) \Omega = \langle 0, \pi \rangle \times \langle 0, 2\pi \rangle$$

$$g) \Omega = \langle 0, 1 \rangle \times \langle 0, 2\pi \rangle$$

$$h) \Omega = \langle -1, 1 \rangle^2$$

$$i) \Omega = \langle -1, 1 \rangle \times \langle 0, 2\pi \rangle$$

$$j) \Omega = \langle 0, \ln(2) \rangle \times \langle 0, 2\pi \rangle$$

$$k) \Omega = \langle 0, 2\pi \rangle \times \langle 0, 2\pi \rangle$$

$$l) \Omega = \langle 0, 1 \rangle \times \langle 0, 2\pi \rangle$$

$$m) \Omega = \langle 0, 1 \rangle \times \langle 0, 2\pi \rangle$$

Riešenie: a) $P = \iint_{\Omega} \sqrt{D_1} du dv = \int_0^3 \int_{-2}^2 du dv = 12$

$$\text{b) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{10} \int_0^{2\pi} r dudv = 20\pi r$$

$$\text{c) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^1 \int_0^{2\pi} rv \sqrt{1+r^2} dudv = \pi r \sqrt{1+r^2}$$

$$\text{d) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^{\pi} r^2 \sin u dudv = 4\pi r^2$$

$$\text{e) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos u dudv = 4\pi r^2$$

$$\text{f) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^{\pi} a^2 \sin u \sqrt{a^2 \cos^2 u + b^2 \sin^2 u} dudv = 2\pi(a^2 + \frac{ab \arccos \frac{a}{b}}{\sin(\arccos \frac{a}{b})})$$

$$\text{g) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^1 au \sqrt{a^2 + 4b^2 u^2} dudv = \frac{\pi a}{6b^2} (\sqrt{(a^2 + 4b^2)^3} - a^2)$$

$$\text{h) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_{-1}^1 \int_{-1}^1 \sqrt{1+u^2+v^2} dudv = \frac{-2\pi}{9} + \frac{4}{3}(\sqrt{3} + \ln(7+4\sqrt{3}))$$

i)

$$P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_{-1}^1 a \cosh u \sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u} dudv =$$

$$= \frac{a\pi}{2e^2 \sqrt{a^2 + b^2}} \left((e^2 - 1) \sqrt{a^2 + b^2} \sqrt{a^2 (e^2 - 1)^2 + b^2 (e^2 + 1)^2} + 4b^2 e^2 \arctan \frac{\sqrt{a^2 + b^2} (e^2 - 1)}{\sqrt{a^2 (e^2 - 1)^2 + b^2 (e^2 + 1)^2}} \right)$$

$$\text{j) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^1 r^2 \cosh^2 u dudv = \frac{3}{2} \pi r^2$$

$$\text{k) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^{2\pi} r(d + r \cos u) dudv = 4\pi^2 r d$$

$$\text{l) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^1 a \sqrt{a^2 u^2 + b^2} dudv = \pi \left(a \sqrt{a^2 + 4b^2} + 4b^2 \ln \frac{a + \sqrt{a^2 + 4b^2}}{2a} \right)$$

$$\text{m) } P = \iint_{\Omega} \sqrt{D_1} dudv = \int_0^{2\pi} \int_0^1 \sqrt{2} v dudv = \sqrt{2} \pi$$