

ÚLOHY O KRIVKÁCH

Príklad 1: Daná je vektorová funkcia

- a) $\mathbf{r}(t) = (\sin t, \cos t, 2 + t^2)$
- b) $\mathbf{r}(t) = (2t, -t^3, t^2 - 1)$
- c) $\mathbf{r}(t) = (\cos t, 1/\cos t, \sin t)$
- d) $\mathbf{r}(t) = (\sin t, \cos t, \tan t)$
- e) $\mathbf{r}(t) = (e^t, e^{2t}, t.e^t)$

Nájdite prvky Frenet - Serretovho repéru (vektory $\mathbf{t}$, $\mathbf{b}$, $\mathbf{n}$) a trojhranu (rovnice priamok - dotyčnica, normála a binormála, rovnice rovín - oskulačná, rektifikačná, normálová) krivky určenej danou vektorovou rovnicou, vyjadrite funkcie prvej a druhej krivosti, a v bode $t = 0$ vypočítajte súradnice bodu krivky P a vektorov $\mathbf{t}$, $\mathbf{b}$, $\mathbf{n}$ a hodnoty 1k , 2k .

Riešenie:

- a) $P = [0, 1, 2]$, $\mathbf{t}(0) = (1, 0, 0)$, $\mathbf{b}(0) = (0, -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $\mathbf{n}(0) = (0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, ${}^1k(0) = \sqrt{5}$, ${}^2k(0) = 0$
- b) $P = [0, 0, -1]$, $\mathbf{t}(0) = (1, 0, 0)$, $\mathbf{b}(0) = (0, 1, 0)$, $\mathbf{n}(0) = (0, 0, -1)$, ${}^1k(0) = 0.5$, ${}^2k(0) = 1.5$
- c) $P = [1, 1, 0]$, $\mathbf{t}(0) = (0, 0, 1)$, $\mathbf{b}(0) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $\mathbf{n}(0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, ${}^1k(0) = \sqrt{2}$, ${}^2k(0) = 0$
- d) $P = [0, 1, 0]$, $\mathbf{t}(0) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $\mathbf{b}(0) = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$, $\mathbf{n}(0) = (0, -1, 0)$, ${}^1k(0) = 0.5$, ${}^2k(0) = -1.5$
- e) $P = [1, 1, 0]$, $\mathbf{t}(0) = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$, $\mathbf{b}(0) = (0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, $\mathbf{n}(0) = (-\frac{3}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{1}{\sqrt{30}})$,
 ${}^1k(0) = \frac{\sqrt{5}}{6\sqrt{6}}$, ${}^2k(0) = -2$

Príklad 2: Nájdite funkciu prvej krivosti ${}^1k(t)$ krivky určenej vektorovou funkciou

- a) $\mathbf{r}(t) = (a \cos t, b \sin t, 0)$, $t \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - elipsa
 - b) $\mathbf{r}(t) = (a/\cos t, b \tan t, 0)$, $t \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - hyperbola
 - c) $\mathbf{r}(t) = (t, pt^2, 0)$, $t \in \mathbf{R}$, $p \in \mathbf{R}$ - parabola
 - d) $\mathbf{r}(t) = (t, a^t, 0)$, $t \in \mathbf{R}$, $a \in \mathbf{R}$, $a \neq 1$ - exponenciála
 - e) $\mathbf{r}(t) = (t, \log_a t, 0)$, $t \in \mathbf{R}^+$, $a \in \mathbf{R}$, $a \neq 1$ - logaritmická krivka
 - f) $\mathbf{r}(t) = (t, \sin t, 0)$, $t \in \mathbf{R}$ - sínusoida
 - g) $\mathbf{r}(t) = (t, \tan t, 0)$, $t \in \langle -\pi/2, \pi/2 \rangle$ - graf funkcie tangens
 - h) $\mathbf{r}(t) = (t, \sinh t, 0)$, $t \in \mathbf{R}$ - reťazovka
 - i) $\mathbf{r}(t) = (a \cos t, b \sin t, vt)$, $t \in \langle 0, 2\pi \rangle$, $a, b \in \mathbf{R}$ - eliptická skrutkovica
 - j) $\mathbf{r}(t) = (r(1-t) \cos 2\pi t, r(1-t) \sin 2\pi t, vt)$, $t \in \langle 0, 2\pi \rangle$, $r, v \in \mathbf{R}$ - kónická skrutkovica
 - k) $\mathbf{r}(t) = (r \sin t \cos 2t, -r \sin t \sin 2t, r \cos t)$, $t \in \langle 0, \pi \rangle$, $r \in \mathbf{R}$ - sférická skrutkovica
 - l) $\mathbf{r}(t) = (-r \sin t \cos t, r \cos^2 t, r \sin t)$, $t \in \langle 0, 2\pi \rangle$, $r \in \mathbf{R}$ - Vivianiho krivka
 - m) $\mathbf{r}(t) = (r \cos t, r \sin t, r \sin t \cos t)$, $t \in \langle 0, 2\pi \rangle$, $r \in \mathbf{R}$ - zvlnená kružnica
- a zistíte, v ktorých bodoch krivky nadobúda prvá krivosť extrémne hodnoty.

Riešenie: a) ${}^1k(t) = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$, hlavné vrcholy $P(0)$, $P(\pi)$ - ${}^1k(t) = ab^{-1}$ - max, vedľajšie

vrcholy $P(\pi/2)$, $P(3\pi/2)$ - ${}^1k(t) = a^{-1}b$ - min

b) ${}^1k(t) = \frac{ab \cos^3 t}{(a^2 \sin^2 t + b^2) \sqrt{a^2 \sin^2 t + b^2}}$, hlavné vrcholy $P(0)$, $P(\pi)$ - ${}^1k(t) = ab^{-2}$ - max

$$c) \quad {}^1k(t) = \frac{2p}{(1+4p^2t^2)\sqrt{1+4p^2t^2}}, \text{ hlavní vrchol } P(0) - {}^1k(t) = 2p - \max$$

$$d) \quad {}^1k(t) = \frac{\ln a}{t^2(1+t^{-2}\ln^2 a)^{3/2}}, \text{ nemá extrém}$$

$$e) \quad {}^1k(t) = \frac{a^t \ln^2 a}{(1+\ln^2 a^{2t})^{3/2}}, \text{ nemá extrém}$$

$$f) \quad {}^1k(t) = \frac{\sin t}{(1+\cos^2 t)^{3/2}}, P(\pi/2) - {}^1k(t) = 1 - \max$$

$$g) \quad {}^1k(t) = \frac{2\sin t \cos^5 t}{(1+\cos^4 t)^{3/2}}, P(\pi/2) - {}^1k(t) = 1 - \max$$

$$h) \quad {}^1k(t) = \frac{\sinh t}{(1+\cosh^2 t)^{3/2}}, P(0) - {}^1k(t) = 1 - \max$$

$$i) \quad {}^1k(t) = \frac{\sqrt{a^2 v^2 \cos^2 t + b^2 v^2 \sin^2 t + a^2 b^2}}{(a^2 \sin^2 t + b^2 \cos^2 t + v^2)^{3/2}},$$

$$P(0), P(\pi) - {}^1k(t) = \frac{a}{b^2 + v^2} - \max, P(\pi/2), P(3\pi/2) - {}^1k(t) = \frac{b}{a^2 + v^2} - \min$$

$$j) \quad {}^1k(t) = \frac{rd_1}{d^3}, d = \sqrt{r^2(2-2t+t^2)+v^2}, d_1 = \sqrt{r^2(3-2t+t^2)+v^2(5-2t+t^2)},$$

$$P(1) - {}^1k(t) = \frac{r\sqrt{2r^2+4v^2}}{(r^2+v^2)^{3/2}} - \max, P(t_1), P(t_2) - {}^1k(t) = \frac{r\sqrt{-r^2+\frac{rv^4}{r^2}-2v^2}}{\left(\frac{-3r^5+rv^4-6r^3v^2+2rv^4}{r(r^2+v^2)}\right)^{3/2}} -$$

$$\min, t_{1,2} = \frac{2r^5+2r^3v^2 \pm \sqrt{2}\sqrt{-r^3(r^2+v^2)(5r^5-rv^4+10r^3v^2)}}{2r^5+2r^3v^2},$$

$$k) \quad {}^1k(t) = \frac{4\sqrt{4(-7+3\cos 2t)^2+(1-6\cos 2t+3\cos 4t)^2+576\cos^2 t \sin^4 t}}{r(\sqrt{(\cos t-3\cos^2 t)^2+4\sin^2 t+(\sin t-3\cos 3t)^2})^3},$$

$$P(0), P(\pi) - {}^1k(t) = \frac{8}{13r}\sqrt{\frac{17}{13}} - \max, P(\pi/2), P(3\pi/2) - {}^1k(t) = \frac{4}{5r}\sqrt{\frac{197}{5}} - \max,$$

$$l) \quad {}^1k(t) = \frac{\sqrt{4(1+\cos^6 t)+\sin^2 t(2+\cos 2t)^2}}{r\sqrt{(1+\cos^2 t)^3}},$$

$$P(0), P(\pi) - {}^1k(t) = \frac{1}{r} - \max, P(\pi/2), P(3\pi/2) - {}^1k(t) = \frac{2\sqrt{2}}{r} - \max$$

$$m) \quad {}^1k(t) = \frac{\sqrt{1+\cos^2 t(-2+\cos 2t)^2+(2+\sin t \cos 2t)^2}}{r\sqrt{(1+\cos^2 2t)^3}},$$

$$P(0), P(\pi) - {}^1k(t) = \frac{1}{r}\sqrt{\frac{3}{4}} - \min, P(\pi/2), P(3\pi/2) - {}^1k(t) = \frac{\sqrt{5}}{r} - \max$$

Príklad 3: Nájdite hodnoty parametra t , ktoré určujú body krivky k s extrémnou druhou krivosťou, ak je krivka hodografom vektorovej funkcie

- a) $\mathbf{r}(t) = (2t, t^2 - 1, 1 - t), t \in \mathbf{R}$
- b) $\mathbf{r}(t) = (\sin 2t, 2\sin t, \cos t)$
- c) $\mathbf{r}(t) = (a \cos t, b \sin t, vt), t \in \langle 0, 2\pi \rangle, a, b \in \mathbf{R}$ - eliptická skrutkovica
- d) $\mathbf{r}(t) = (r(1-t) \cos 2\pi t, r(1-t) \sin 2\pi t, vt), t \in \langle 0, 1 \rangle, r, v \in \mathbf{R}$ - kónická skrutkovica
- e) $\mathbf{r}(t) = (r \sin \pi t \cos 2\pi t, -r \sin \pi t \sin 2\pi t, r \cos \pi t), t \in \langle 0, 1 \rangle, r \in \mathbf{R}$ - sférická skrutkovica
- f) $\mathbf{r}(t) = (-r \sin t \cos t, r \cos^2 t, r \sin t), t \in \langle 0, 2\pi \rangle, r \in \mathbf{R}$ - Vivianiho krivka
- g) $\mathbf{r}(t) = (r \cos t, r \sin t, r \sin t \cos t), t \in \langle 0, 2\pi \rangle, r \in \mathbf{R}$ - zvlnená kružnica

Riešenie: a) ${}^2k(t) = 0, t \in \mathbf{R}$

b) ${}^2k(t) = 0, t \in \mathbf{R}$

c) ${}^2k(t) = \frac{abv}{a^2v^2 \cos^2 t + b^2v^2 \sin^2 t + a^2b^2}, P(0), P(\pi) - {}^2k(t) = \frac{abv}{a^2v^2 + a^2b^2} - \min,$
 $P(\pi/2), P(3\pi/2) - {}^2k(t) = \frac{abv}{b^2v^2 + a^2b^2} - \max$

d) ${}^2k(t) = \frac{7-2t+t^2}{r^2(3-2t+t^2)+v^2(5-2t+t^2)}, P(0) - {}^2k(t) = \frac{7}{3r^2+5v^2} - \min$

e) ${}^2k(t) = \frac{-11 \cos t + 3 \cos 3t + 4(\sin 2t - 6 \sin 4t + 3 \sin 6t)}{r((-7+3 \cos 2t)^2 + 4(1-6 \cos 2t + 3 \cos 4t)^2 + 576 \sin^6 t \cos^2 t)},$
 $P(0) - \max, P(\pi) - \min, {}^2k(t) = \frac{\pm 4}{17r}$

f) ${}^2k(t) = \frac{6 \cos t}{r(4(1+\cos^6 t) + \sin^2 t(2+\cos 2t)^2)}, P(0) - \max, P(\pi) - \max, {}^2k(t) = \frac{\pm 3}{4r}$

g) ${}^2k(t) = \frac{12 \cos 2t}{r(4(-7+3 \cos 2t)^2 + (1-6 \cos 2t + 3 \cos 4t)^2 + 576 \cos^2 t \sin^6 t)}$

$P(0) - \max, P(\pi) - \max, {}^2k(t) = \frac{12}{17r}, P(\pi/2), P(3\pi/2) - \min, {}^2k(t) = 0$

Príklad 4: Ukážte, či sú krivky určené vektorovými funkciami

- a) $\mathbf{r}(t) = (r \cos t, r \sin t, vt), t \in \langle 0, 2\pi \rangle, r \in \mathbf{R}$ - cylindrická skrutkovica
 - b) $\mathbf{r}(t) = (a \cos t, b \sin t, vt), t \in \langle 0, 2\pi \rangle, a, b \in \mathbf{R}$ - eliptická skrutkovica
 - c) $\mathbf{r}(t) = (r(1-t) \cos t, r(1-t) \sin t, vt), t \in \langle 0, 2\pi \rangle, r, v \in \mathbf{R}$ - kónická skrutkovica
 - d) $\mathbf{r}(t) = (r \sin t \cos 2t, -r \sin t \sin 2t, r \cos t), t \in \langle 0, \pi \rangle, r \in \mathbf{R}$ - sférická skrutkovica
- krivkami konštantného spádu.

Riešenie: a) $\frac{{}^1k(t)}{{}^2k(t)} = \frac{r}{v}$

b) $\frac{{}^1k(t)}{{}^2k(t)} = \frac{1}{abv} \left(\frac{a^2v^2 \cos^2 t + b^2v^2 \sin^2 t + a^2b^2}{a^2 \sin^2 t + b^2 \cos^2 t + v^2} \right)^{\frac{3}{2}},$ nie je krivka konštantného spádu

c) $\frac{{}^1k(t)}{{}^2k(t)} = \frac{d_1^3}{rd^3} = \frac{1}{r} \left(\frac{d_1}{d} \right)^3$

$$d) \frac{{}^1k(t)}{{}^2k(t)} = \frac{1}{r} \left(\frac{d_2}{d_3} \right)^3$$

Príklad 5: Nájdite vektorovú funkciu kriviek, ktoré sú ekvidistantami krivky určenej vektorovou funkciou

- a) $\mathbf{r}(t) = (t, at^2 + bt + c, 0), t \in \mathbf{R}, a, b, c \in \mathbf{R}$
- b) $\mathbf{r}(t) = (t, \cos t, 0), t \in \langle 0, 2\pi \rangle$
- c) $\mathbf{r}(t) = (t, \ln t, 0), t \in (0, \infty)$
- d) $\mathbf{r}(t) = (r \cos t, r \sin t, vt), t \in \langle 0, 2\pi \rangle, r, v \in \mathbf{R}$
- e) $\mathbf{r}(t) = ((1-t) \cos t, 0, vt), t \in \langle 0, 2\pi \rangle, v \in \mathbf{R}$

Riešenie: a) $\mathbf{p}(t) = \left(t \pm k \cdot \frac{2at}{\sqrt{1+(2at+b)^2}}, at^2 + bt \pm k \cdot \frac{1}{\sqrt{1+(2at+b)^2}}, 0 \right), k \in \mathbf{R}$

b) $\mathbf{p}(t) = \left(t \pm k \cdot \frac{\sin t}{\sqrt{1+\sin^2 t}}, \cos t \pm k \cdot \frac{-1}{\sqrt{1+\sin^2 t}}, 0 \right), k \in \mathbf{R}$

c) $\mathbf{p}(t) = \left(t \pm k \cdot \frac{1}{\sqrt{1+t^2}}, \ln t \pm k \cdot \frac{-t}{\sqrt{1+t^2}}, 0 \right), k \in \mathbf{R}$

d) $\mathbf{p}(t) = (\cos t(r \pm k \cdot \sqrt{r^2 + v^2}), \sin t(r \pm k \cdot \sqrt{r^2 + v^2}), vt), k \in \mathbf{R}$

e) $\mathbf{p}(t) = \left((1-t) \cos t \pm \frac{vk}{bd}, 0, vt \pm \frac{k(\cos t + (1-t) \sin t)}{bd} \right), k \in \mathbf{R}$

$$d = \sqrt{(\cos t + (1-t) \sin t)^2 + v^2}, b = 2 \sin t + (t-1) \cos t$$

Príklad 6: Nájdite vektorovú funkciu kvadratickej krivky definovanej na intervale $\langle 0, 1 \rangle$, ktorej začiatočným bodom je bod $P(0)$ a koncovým bodom bod $P(1)$, pričom vektor $\mathbf{d}(t)$ je vektorom dotyčnice krivky v bode $P(t)$

- a) $P(0) = [2, 2, 0], P(1) = [0, 2, 2], \mathbf{d}(0) = (2, 3, -2)$
- b) $P(0) = [4, 0, 0], P(1) = [0, 4, 0], \mathbf{d}(1) = (2, 1, 3)$

Riešenie: a) $\mathbf{r}(t) = (-4t^2 + 2t + 2, -3t^2 + 3t + 2, 4t^2 - 2t)$

b) $\mathbf{r}(t) = (6t^2 - 10t + 4, -3t^2 + 7t, -3t^2 - 3t)$

Príklad 7: Nájdite vektorovú funkciu kubické krivky definovanej na intervale $\langle 0, 1 \rangle$, ktorej začiatočným bodom je bod $P(0)$ a koncovým bodom bod $P(1)$, pričom vektory $\mathbf{d}(0)$ a $\mathbf{d}(1)$ sú vektormi dotyčnic krivky v príslušných bodoch

- a) $P(0) = [2, 1, 0], P(1) = [1, 4, 2], \mathbf{d}(0) = (0, 0, 2), \mathbf{d}(1) = (0, 2, 0)$
- b) $P(0) = [0, 0, 4], P(1) = [0, 0, 0], \mathbf{d}(0) = (1, 1, -1), \mathbf{d}(1) = (-1, -1, 0)$

Riešenie: a) $\mathbf{r}(t) = (2t^3 - 3t^2 + 2, -4t^3 + 7t^2 + 1, -2t^3 + 2t^2 + 2t)$

b) $\mathbf{r}(t) = (-t^2 + t, -2t^3 + t^2 + t, 8t^3 - 12t^2 + 4)$

Príklad 8: Nájdite vektorovú funkciu kubické krivky definovanej na intervale $\langle 0, 1 \rangle$, ktorej začiatočným bodom je bod $P(0)$ a koncovým bodom bod $P(1)$, pričom vektor $\mathbf{d}(0)$ je vektorom dotyčnice a vektor $\mathbf{b}(0)$ vektorom binormály krivky v bode $P(0)$

- a) $P(0) = [1, 0, 0], P(1) = [0, 1, 1], \mathbf{d}(0) = (1, 0, 0), \mathbf{b}(0) = (0, 0, -1)$
- b) $P(0) = [2, 2, 0], P(1) = [0, 0, 2], \mathbf{d}(0) = (0, 1, 0), \mathbf{b}(0) = (0, 0, 1)$

Riešenie: a) $\mathbf{r}(t) = (at^3 + (-2-a)t^2 + t + 1, 1\frac{1}{2}t^3 - \frac{1}{2}t^2, t^3)$

b) $\mathbf{r}(t) = (-1\frac{1}{2}t^3 - \frac{1}{2}t^2 + 2, at^3 - (a+3)t^2 + t + 2, 2t^3)$