Rules of Differentiation

Aim

To introduce the rules of differentiation.

Learning Outcomes

At the end of this section you will be able to:

- Identify the different rules of differentiation,
- Apply the rules of differentiation to find the derivative of a given function.

The basic rules of differentiation are presented here along with several examples. Remember that if \( y = f(x) \) is a function then the derivative of \( y \) can be represented by \( \frac{dy}{dx} \) or \( y' \) or \( \frac{df}{dx} \). The basic rules of differentiation, as well as several common results, are presented in the back of the log tables on pages 41 and 42.

Rule 1: The Derivative of a Constant.

The derivative of a constant is zero.

Rule 2: The General Power Rule.

The derivative of \( x^n \) is \( nx^{n-1} \).

Example 1

Differentiate \( y = x^4 \).

If \( y = x^4 \) then using the general power rule, \( \frac{dy}{dx} = 4x^3 \).

Rule 3: The Derivative of a Constant times a Function.

The derivative of \( kf(x) \), where \( k \) is a constant, is \( kf'(x) \).

Example 2

Differentiate \( y = 3x^2 \).

In this case \( f(x) = x^2 \) and \( k = 3 \), therefore the derivative is \( 3 \times 2x^1 = 6x \).

Rule 4: The Derivative of a Sum or a Difference.

If \( f(x) = h(x) \pm g(x) \), then \( \frac{df}{dx} = \frac{dh}{dx} \pm \frac{dg}{dx} \).
Example 3
Differentiate \( f(x) = 3x^2 - 7x \).

In this case \( k(x) = 3x^2 \) and \( g(x) = 7x \) and so \( \frac{dk}{dx} = 6x \) and \( \frac{dg}{dx} = 7 \). Therefore, \( \frac{df}{dx} = 6x - 7 \).

Rule 5: The Product Rule.
The derivative of the product \( y = u(x)v(x) \), where \( u \) and \( v \) are both functions of \( x \) is
\[
\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.
\]

Example 4
Differentiate \( f(x) = (6x^2 + 2x)(x^3 + 1) \).

Let \( u(x) = 6x^2 + 2x \) and \( v(x) = x^3 + 1 \). Therefore,
\[
\frac{du}{dx} = 12x + 2 \quad \text{and} \quad \frac{dv}{dx} = 3x^2.
\]

Therefore using the formula for the product rule,
\[
\frac{df}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.
\]
we get,
\[
\frac{df}{dx} = (6x^2 + 2x)(3x^2) + (x^3 + 1)(12x + 2),
= 18x^4 + 6x^3 + 12x^4 + 2x^3 + 12x + 2,
= 30x^4 + 8x^3 + 12x + 2.
\]

Rule 6: The Quotient Rule.
The derivative of the quotient \( f(x) = \frac{u(x)}{v(x)} \), where \( u \) and \( v \) are both function of \( x \) is
\[
\frac{df}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}.
\]

Example 5
Differentiate \( f(x) = \frac{x^2 + 7}{3x - 1} \).

Let \( u(x) = x^2 + 7 \) and \( v(x) = 3x - 1 \). Differentiate these to get \( \frac{du}{dx} = 2x \) and \( \frac{dv}{dx} = 3 \).
Now using the formula for the quotient rule we get,
\[
\frac{df}{dx} = \frac{(3x - 1)(2x) - (x^2 + 7)(3)}{(3x - 1)^2},
\]
\[
= \frac{6x^2 - 2x - 3x^2 - 21}{(3x - 1)^2},
\]
\[
\Rightarrow \frac{df}{dx} = \frac{3x^2 - 2x - 21}{(3x - 1)^2}.
\]

**Rule 7: The Chain Rule.**

If \( y \) is a function of \( u \), i.e. \( y = f(u) \), and \( u \) is a function of \( x \), i.e. \( u = g(x) \) then the derivative of \( y \) with respect to \( x \) is

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.
\]

**Example 6**

Differentiate \( y = (x^2 - 5)^4 \).

Let \( u = x^2 - 5 \), therefore \( y = u^4 \).

\[
\Rightarrow \frac{du}{dx} = 2x \quad \text{and} \quad \Rightarrow \frac{dy}{du} = 4u^3.
\]

Using the chain rule we then get

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},
\]
\[
= 4u^3 \times 2x,
\]
\[
= 4(x^2 - 5)^3 \times 2x,
\]
\[
= 8x(x^2 - 5)^3.
\]

**Related Reading**
