Rules of Differentiation

Aim

To introduce the rules of differentiation.

Learning Outcomes

At the end of this section you will be able to:

- Identify the different rules of differentiation,
- Apply the rules of differentiation to find the derivative of a given function.

The basic rules of differentiation are presented here along with several examples. Remember that if $y = f(x)$ is a function then the derivative of y can be represented by $\frac{dy}{dx}$ or y' or f' or $\frac{df}{dx}$. The basic rules of differentiation, as well as several common results, are presented in the back of the log tables on pages 41 and 42.

Rule 1: The Derivative of a Constant.

The derivative of a constant is zero.

Rule 2: The General Power Rule.

The derivative of x^n is nx^{n-1}.

Example 1

Differentiate $y = x^4$.

If $y = x^4$ then using the general power rule, $\frac{dy}{dx} = 4x^3$.

Rule 3: The Derivative of a Constant times a Function.

The derivative of $kf(x)$, where k is a constant, is $kf'(x)$.

Example 2

Differentiate $y = 3x^2$.

In this case $f(x) = x^2$ and $k = 3$, therefore the derivative is $3 \times 2x^1 = 6x$.

Rule 4: The Derivative of a Sum or a Difference.

If $f(x) = h(x) \pm g(x)$, then $\frac{df}{dx} = \frac{dh}{dx} \pm \frac{dg}{dx}$.

Example 3

Differentiate $f(x) = 3x^2 - 7x$.

In this case $k(x) = 3x^2$ and $g(x) = 7x$ and so $\frac{dk}{dx} = 6x$ and $\frac{dg}{dx} = 7$. Therefore, $\frac{df}{dx} = 6x - 7$.

Rule 5: The Product Rule.

The derivative of the product $y = u(x)v(x)$, where u and v are both functions of x is

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.$$

Example 4

Differentiate $f(x) = (6x^2 + 2x)(x^3 + 1)$.

Let $u(x) = 6x^2 + 2x$ and $v(x) = x^3 + 1$. Therefore,

$$\frac{du}{dx} = 12x + 2 \qquad \text{and} \qquad \frac{dv}{dx} = 3x^2.$$

Therefore using the formula for the product rule,

$$\frac{df}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.$$

we get,

$$\begin{aligned} \frac{df}{dx} &= (6x^2 + 2x)(3x^2) + (x^3 + 1)(12x + 2), \\ &= 18x^4 + 6x^3 + 12x^4 + 2x^3 + 12x + 2, \\ &= 30x^4 + 8x^3 + 12x + 2. \end{aligned}$$

Rule 6: The Quotient Rule.

The derivative of the quotient $f(x) = \frac{u(x)}{v(x)}$, where u and v are both function of x is

$$\frac{df}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}.$$

Example 5

Differentiate $f(x) = \frac{x^2 + 7}{3x - 1}$.

Let $u(x) = x^2 + 7$ and $v(x) = 3x - 1$. Differentiate these to get $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3$. Now using the formula for the quotient rule we get,

$$\begin{aligned}\frac{df}{dx} &= \frac{(3x-1)(2x)-(x^2+7)(3)}{(3x-1)^2},\\ &= \frac{6x^2-2x-3x^2-21}{(3x-1)^2},\\ \Rightarrow \frac{df}{dx} &= \frac{3x^2-2x-21}{(3x-1)^2}.\end{aligned}$$

Rule 7: The Chain Rule.

If y is a function of u, i.e. $y = f(u)$, and u is a function of x, i.e. $u = g(x)$ then the derivative of y with respect to x is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Example 6

Differentiate $y = (x^2-5)^4$.

Let $u = x^2 - 5$, therefore $y = u^4$.

$\Rightarrow \frac{du}{dx} = 2x$ and $\Rightarrow \frac{dy}{du} = 4u^3$.

Using the chain rule we then get

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx},\\ &= 4u^3 \times 2x,\\ &= 4(x^2-5)^3 \times 2x,\\ &= 8x(x^2-5)^3.\end{aligned}$$

Related Reading

Adams, R.A. 2003. *Calculus: A Complete Course.* 5^{th} Edition. Pearson Education Limited.

Morris, O.D., P. Cooke. 1992. *Text & Tests 4.* The Celtic Press.