DIFFERENCE EQUATIONS – BASIC DEFINITIONS AND PROPERTIES

Difference equations can be viewed either as a discrete analogue of differential equations, or independently. They are used for approximation of differential operators, for solving mathematical problems with recurrences, for building various discrete models, etc.

Definition 1. The equation:

(1) $u_{n+k} = f(u_{n+k-1}, u_{n+k-2}, \ldots, u_n)$,

for a given function f and unknown quantities u_i, $i = 0, 1, \ldots$ is called a *difference equation of order k*.

When the equation is of the form:

(2) $a_k u_{n+k} + a_{k-1} u_{n+k-1} + \ldots + a_0 u_n = b$, $a_0 \, a_k \neq 0$

it is called *a linear difference equation*.

According to whether the coefficients and the right hand side of the equation depend on *n* or not, it is called an equation with variable or constant coefficients respectively.

When the right hand side $b \neq 0$, the equation (2) is *non-homogeneous*, while for $b = 0$, i.e.

(3) $a_k u_{n+k} + a_{k-1} u_{n+k-1} + \ldots + a_0 u_n = 0$

the equation is called a *linear homogeneous difference equation*.

Definition 2. *Solution* of the equation (1) (or (2), respectively) is called every number sequence $v = (v_n)_{n=0}^{\infty}$, whose random *k*+1 consecutive members, substituted in the equation, transform it into a number equality.

The combination of all possible solutions forms the *general solution* of the equation, while every separate solution is its *particular solution*.

Normally the general solution of a difference equation of order *k* depends on *k* random constants, which can be simply defined for example by assigning *k* with initial conditions $u_0, u_1, \ldots, u_{k-1}$.

Example. The equation $u_{n+2} = u_n + u_{n+1}$ is a linear homogeneous difference equation of the second order. If we assign two initial conditions by the equalities $u_0 = 1$, $u_1 = 1$, the sequence $u = (u_n)_{n=0}^{\infty}$, which is obtained from that equation, is the well-known Fibonacci sequence. It is easy to calculate that it is as follows:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...

This sequence is a particular solution of the given equation under the assigned initial conditions.

Properties of the solutions of linear difference equations with constant coefficients

Property 1^0). If the number sequences $v^{(1)}$ and $v^{(2)}$ are solutions of the homogeneous equation (3) and α, β are random numbers, then their linear combination $\alpha v^{(1)} + \beta v^{(2)}$ is also a solution of (3).

Definition 3. Let the sequences

(4) $$v^{(1)} = \left\{ v_n^{(1)} \right\}_{n=0}^{\infty}, \quad v^{(2)} = \left\{ v_n^{(2)} \right\}_{n=0}^{\infty}, \ldots, \quad v^{(m)} = \left\{ v_n^{(m)} \right\}_{n=0}^{\infty}$$

be m solutions of the homogeneous equation (3). We will say that they are *linearly independent* if from the equation $\sum_{j=0}^{m} c_j v_n^{(j)} = 0$, which is satisfied for all $n = 0, 1, 2, \ldots$, there follows that the constants $c_j = 0, \ \ j = 1, 2, \ldots, m$. Otherwise they are *linearly dependent.*

Property 2^0). The solutions (4) are linearly independent then and only then when the determinant, which is composed from the first m members of their corresponding sequences, is different from zero, i.e.

$$\begin{vmatrix} v_0^{(1)} & v_0^{(2)} & \ldots & v_0^{(m)} \\ v_1^{(1)} & v_1^{(2)} & \ldots & v_1^{(m)} \\ v_2^{(1)} & v_2^{(2)} & \ldots & v_2^{(m)} \\ \ldots & & \ldots & \\ v_{m-1}^{(1)} & v_{m-1}^{(2)} & \ldots & v_{m-1}^{(m)} \end{vmatrix} \neq 0$$

Property 3^0). Let the sequences $v^{(1)} = \left\{ v_n^{(1)} \right\}_{n=0}^{\infty}, \quad v^{(2)} = \left\{ v_n^{(2)} \right\}_{n=0}^{\infty}, \ldots, \quad v^{(k)} = \left\{ v_n^{(k)} \right\}_{n=0}^{\infty}$ be k solutions of the homogeneous equation (3). Then each solution of (3) can be represented as their linear combination.

Property 4^0). The general solution of (2) is a sum from the general solution $\overline{v}$ of the corresponding homogeneous equation (3) and any particular solution v^* of the non-homogeneous equation (2):

(5) $$u = \overline{v} + v^*.$$

Definition 3. The algebraic equation

(6) $$\rho(z) = a_k z^k + a_{k-1} z^{n-1} + \ldots + a_0 = 0$$

is called a *characteristic equation* of (2).

Let the coefficients $a_i, \ \ i = 0, 1, \ldots, k$ be real numbers. We denote the roots of (6) by $z_1, z_2, \ldots, z_k$ respectively.

Property 5^0). If all the roots of the characteristic equation $\rho(z)=0$ are simple, i.e. $z_i \neq z_j$ for $i \neq j$, then the sequences $\left\{ z_1^n \right\}$, $\left\{ z_2^n \right\}$, ..., $\left\{ z_k^n \right\}$ are linearly independent solutions of the homogeneous difference equation (3) and its general solution $\bar{v}$ is represented as their linear combination $\bar{v} = \left\{ \overline{v_n} \right\}_{n=0}^{\infty} = \left\{ \sum_{i=1}^{k} c_i\, z_i^n \right\}_{n=0}^{\infty}$, where $c_i = 1, 2, ..., k$ are random constants.

Property 6^0). If z is a real root of the characteristic equation $\rho(z)=0$ with multiplicity s, then the sequences $\left\{ z^n \right\}_{n=0}^{\infty}$, $\left\{ n\, z^n \right\}_{n=0}^{\infty}$, $\left\{ n^2\, z^n \right\}_{n=0}^{\infty}$, ..., $\left\{ n^{s-1}\, z^n \right\}_{n=0}^{\infty}$, are linearly independent solutions of the difference equation (3).

Property 7^0). If $z = r\,e^{i\theta}$, $\bar{z} = r\,e^{-i\theta}$ are two complexly conjugated roots of the characteristic equation $\rho(z)=0$, then $\left\{ r^n \cos n\theta \right\}_{n=0}^{\infty}$, $\left\{ r^n \sin n\theta \right\}_{n=0}^{\infty}$ are solutions of the homogeneous difference equation (3).

Property 8^0). Let z and $\bar{z}$ be two complexly conjugated roots of the characteristic equation $\rho(z)=0$ with multiplicity s. Then the sequences

$$\left\{ r^n \cos n\theta \right\}_{n=0}^{\infty}, \quad \left\{ n\, r^n \cos n\theta \right\}_{n=0}^{\infty}, \quad \left\{ n^2 r^n \cos n\theta \right\}_{n=0}^{\infty}, \ldots, \left\{ n^{s-1} r^n \cos n\theta \right\}_{n=0}^{\infty},$$

$$\left\{ r^n \sin n\theta \right\}_{n=0}^{\infty}, \quad \left\{ n\, r^n \sin n\theta \right\}_{n=0}^{\infty}, \quad \left\{ n^2 r^n \sin n\theta \right\}_{n=0}^{\infty}, \ldots, \left\{ n^{s-1} r^n \sin n\theta \right\}_{n=0}^{\infty}$$

are linearly independent solutions of the difference equation (3).

Property 9^0). If z is an s – multiple root of the characteristic equation (6) and the right hand side of the non-homogeneous equation (2) is $b = b_n = \left\{ (\alpha_0 n^s + \alpha_1 n^{s+1} + \ldots + \alpha_m n^{s+m})\; z^n \right\}_{n=0}^{\infty}$, then (2) has a particular solution of the following type: $v^* = \left\{ (\beta_0 n^s + \beta_1 n^{s+1} + \ldots + \beta_m n^{s+m})\; z^n \right\}_{n=0}^{\infty}$. In case z is not a root of (3), we assume that $s = 0$.

In the common case with $b = b(n)$, we are looking for the particular solution of (2) through the method of Lagrange by varying the constants, i.e. in the form: $u^* = \sum_{j=1}^{k} c_j(n)\, z_j^n$.

Definition 4. The homogeneous difference equation (3) is called <u>*stable by initial data*</u> if there exists a constant C >0, which does not depend on its solution $u = \left\{ u_n \right\}_{n=0}^{\infty}$ and is such that for each natural n is valid the following inequality:

(7) $$\left| u_n \right| \leq C \max_{0 \leq j \leq k-1} \left| u_j \right| .$$

Note. It is obvious from property 4^0) that the homogeneous equation is unstable, i.e. if there is a solution which grows indefinitely, then the non-homogeneous equation will be unstable too.

Theorem 1. The necessary and sufficient condition for stability of the homogeneous equation (3) (by initial data) is for the absolute value of the simple roots of the respective characteristic equation to be smaller or equal to one, and the absolute value of the multiple roots to be strictly smaller than one.

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