Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

STOCHASTIC APPROXIMATION PROCEDURE FOR ISOLATED POPULATION IN FAST OSCILLATED RANDOM ENVIRONMENT

CARKOVS Jevgeņijs (LV), ŠADURSKIS Kārlis (LV), GOLDŠTEINE Jolanta (LV)

Abstract. The paper deals with a logistic type population mathematical model given in a form of Markov impulsive differential equation controlled by a fast oscillating ergodic Poisson process. Applying the stochastic approximation procedure, we construct and analyse ordinary differential equations for the mean value of population size and stochastic Ito differential equation for random deviations on the mean values.

Key words: stochastic modelling, logistic equations, stochastic approximation

Mathematics Subject Classification: Primary 92B05,60H30,08A72; Secondary 60G57, 37H10

1 Introduction

In classical mathematical models of population dynamics, as a rule, an assumption is made that external conditions are constant. Therefore, the logistic model [1] is the most popular, for example, for describing the dynamics of an isolated population. It assumes that for any time moment $t \geq 0$ for a sufficiently small time interval Δ the increment $\Delta x(t) = x(t+\Delta) - x(t)$ of the biomass $x(t)$ of the population can be represented in the form of an asymptotic equality:

$$\Delta x(t) = rx(t)(1 - K^{-1}x(t))\Delta - mx(t)\Delta + o(\Delta)\,, \tag{1.1}$$

where K is the capacity of the habitat, $r(1-K^{-1}x)$ is fertility, m is the death rate. In this case, we can divide both sides of (1.1) by Δ , approach to the limit as $\Delta \to 0$ and use the ordinary differential equation

$$\frac{dx}{dt} = rx(1 - K^{-1}x) - mx \tag{1.2}$$

as a mathematical model.

The main advantage of this model is the ability to solve the equation explicitly and to make sure that the behaviour of the population depends only on the sign of the difference $r-m$. If this difference is negative, then the population is bound to become extinct, if it is positive, then the population size increases and monotonically approaches the value $Kr^{-1}(r-m)$. Unfortunately, this forecast does not always come true: in the real population environment there are random irregular perturbations, and the fertility $r(1-K^{-1}x)$ and mortality mx in the formula (1.1) actually are regression lines. Therefore, one cannot rely on the conclusions made only based on the analysis of the parameters r and m of these regression lines. Under the influence of random perturbations the population may leave the neighbourhood of a stable equilibrium $Kr^{-1}(r-m)$ and may enter the neighbourhood of zero and, in fact, disappear. This leads to the necessity for specifying increments in (1.1) in the form of the time-dependent random functions dependent on the phase coordinate x and the time interval Δ. Moreover, it is desirable to randomize the model retaining the predictability property of mathematical model of population dynamics. Therefore, papers have been published (see, for example, [3- 8] and references there) in which increments of a logistic type population are given in the form

$$\Delta x(t)=rx(t)(1-K^{-1}x(t))\Delta-mx(t)\Delta+x(t)B(x(t))(w(t+\Delta)-w(t)) \tag{1.3}$$

where $B(x)=b+b_1x$, and $w(t)$ is the standard Brownian motion process, that is, a process with independent normally $N(0,\Delta)$ distributed increments $w(t+\Delta)-w(t)$. The formula for increments (1.3) allows us to propose a stochastic differential equation as a randomized logistic model

$$dx(t)=rx(t)(1-K^{-1}x(t))dt-mx(t)dt+x(t)B(x(t))dw(t) \tag{1.4}$$

The proposed mathematical model has the predictability property because the equation (1.4) determines the Markov process [5]. Using the apparatus of modern stochastic analysis, one can draw conclusions about the qualitative behaviour of the trajectories of solutions (1.4) (the existence and asymptotic stability of the stochastic equilibrium, the tendency of approaching zero, the time of staying in a given interval, etc.). A shortcoming of the model (1.4) is the inability to simulate an abrupt change in the population size because any trajectory of the solution of equation (1.4) with the probability of one is continuous. This feature precludes the use this model (1.4) for populations subject to possible losses at random time moments (selection such as fishing or hunting, predator attacks, etc.). This model (1.4) similarly cannot be used for small populations, since the trajectory $x(t)$ must become discontinuous at the moment of death of even a single individual. We analyse a model with discontinuous trajectories whose dynamic characteristics depend on a piecewise constant ergodic Poisson process $y(t)$ with values in some set $U\subset\mathbf{R}^m$, invariant probability measure $\pi(du)$ and switching time moments $\mathbf{T}=\{\tau_k,k\in\mathbf{N}\}$. We propose that the initial size of the population at the time moment $t=0$ is equal to $x(0)=x_0$. At each interval $[\tau_{k-1},\tau_k),k\in N$, the aggregate biomass $\{x(t),\mathrm{t}\geq 0\}$ of individuals of the population varies logistically

$$\frac{dx(t)}{dt}=rx(t)(1-K^{-1}x(t))-mx(t) \tag{1.5}$$

where ε is a small positive parameter, and at each moment $t \in \mathbf{T}$ the mass of the population goes from the point $x(t-) = \lim_{s \downarrow 0} x(t-s)$ to the point

$$x(t) = x(t-) + \varepsilon x(t-) b(y(\varepsilon^{-1} t-), x(t-)) . \tag{1.6}$$

We briefly describe the stochastic approximation procedure proposed in [12] for the dynamical system with Markov switching, and we apply these results to analyse the dynamics of the logistic model given by equations of the form (1.5) - (1.6) for sufficiently small ε . We have already used this method in [11] to study a simpler impulsive Markov dynamical system given by a stochastic equation of the form

$$dx(t) = rx(t)(1 - K^{-1} x(t))dt - \varepsilon m \int_0^1 x(t) c(u) \mu(\varepsilon^{-1}, dt, du)$$

where $\mu(\varepsilon^{-1}, dt, du)$ is Poisson measure with parameter $\varepsilon^{-1} dtdu$.

2 Stochastic approximation procedure for Markov impulsive dynamical system

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a probability space, let $\mathrm{Y} := \{y_k, k \in \mathbf{N}\}$ be the ergodic Markov chain on the measurable space $(U, \Sigma_U), U \subset \mathbf{R}^m$, with transition probability $p(y, du)$ and invariant measure $\pi(du)$, let $\mu(\lambda, dt, du)$ be the Poisson type measure [5] on the measurable space $(U \times \mathbf{R}, \Sigma_{U \times \mathbf{R}})$ with parameter $\lambda p(y, du)dt$. The stochastic differential equation

$$dy = \int_U (u - y) \mu(\lambda, dt, du) \tag{2.1}$$

defines [5] the ergodic piecewise constant Markov process $\{y(t), t \geq 0\}$ on (U, Σ_U) with the weak infinitesimal operator

$$Qv(y) = \lambda \left[\int_U v(u) p(y, du) - v(y) \right] \tag{2.2}$$

and the invariant measure $\pi(dy)$. This means that there exists a sequence of i.i.d. random variables $\mathrm{D} := \{\delta_k, k \in \mathbf{N}\}$ such that $y(t) \equiv y(\tau_{k-1})$ for any $k \in \mathbf{N}, t \in [\tau_{k-1}, \tau_k)$, where $\tau_k = \sum_{n=1}^{k} \delta_n, \tau_0 = 0$, and $y(\tau_k) = y_k$. This sequence is independent on Markov chain Y and distributed exponentially with parameter λ . Using the above notations and definitions we define impulsive Markov dynamical system with phase coordinate $\{x_\varepsilon(t), t \geq 0\}$ in $\mathbf{R}$ as follows:

$$\tau_{k-1} < t < \tau_k, k \in N: \ \frac{dx_\varepsilon(t)}{dt} = F(x_\varepsilon(t)), \tag{2.3}$$

$$t \in \mathrm{T}: \quad x_\varepsilon(t) = x_\varepsilon(t-) + \varepsilon G(x_\varepsilon(t-), y_\varepsilon(t-)) \tag{2.4}$$

where $y_\varepsilon(t) = y(\varepsilon^{-1}t)$, and $\varepsilon \in (0, \varepsilon_0)$ is a small parameter, which we will use for approximative analysis of dynamical system (2.1) - (2.2). The weak infinitesimal operator [9] for the Markov process $\{y_\varepsilon(t), x_\varepsilon(t), t \ge 0\}$ may be presented in a following form:

$$L(\varepsilon)v(x,y) = F(x)\frac{\partial}{\partial x}v(x,y) + \varepsilon^{-1}Qv(x,y) + \\ + \varepsilon^{-1}\lambda\left[v(x+\varepsilon G(x,y),y) - v(x,y)\right] \qquad (2.5)$$

The first approximative dynamical system for (2.3) - (2.4) is an ordinary differential equation

$$\frac{d}{dt}\bar{x}(t) = f(\bar{x}(t)) \qquad (2.6)$$

where $f(x) := \int_U f_0(x,y)\pi(dy),\ f_0(x,y) = F(x) + \lambda G(x,y)$. By definition [9] of the weak infinitesimal operator (2.2) the equation

$$QV(x,y) = f(x) - f_0(x,y) \qquad (2.7)$$

has the solution $V(x,y) = \Pi f_0(x,y)$ that satisfies an identity

$$\int_U \Pi f_0(x,y)\pi(dy) \equiv 0 \qquad (2.8)$$

Now we can take a limit

$$\lim_{\varepsilon\to 0} L(\varepsilon)\left[v(x) + \varepsilon\Pi f_0(x,y)\frac{\partial}{\partial x}v(x)\right] = \left\{\lim_{\varepsilon\to 0} F(x,y)\left[\frac{\partial}{\partial x}v(x) + \varepsilon\Pi f_0(x,y)\frac{\partial}{\partial x}v(x)\right] + \right.$$

$$+\lim_{\varepsilon\to 0}\varepsilon^{-1}Q\left[v(x) + \varepsilon\Pi f_0(x,y)\frac{\partial}{\partial x}v(x)\right] + \lim_{\varepsilon\to 0}\varepsilon^{-1}\lambda\left[v(x+\varepsilon G(x,y)) - v(x)\right] +$$

$$\left. + \lambda\lim_{\varepsilon\to 0}\left[\Pi f_0(x+\varepsilon G(x,y)) - \Pi f_0(x,y)\right]\frac{\partial}{\partial x}v(x)\right\} =$$

$$= F(x)\frac{\partial}{\partial x}v(x) + \lambda G(x,y)\frac{\partial}{\partial x}v(x) + Q\Pi f_0(x,y)\frac{\partial}{\partial x}v(x) = f(x)\frac{\partial}{\partial x}v(x)$$

which provides

$$\lim_{\varepsilon\to 0}\{v(x) + \varepsilon\Pi f_0(x,y), L(\varepsilon)[v(x) + \varepsilon\Pi f_0(x,y)]\} = \left\{v(x), f(x)\frac{d}{dx}v(x)\right\} \qquad (2.9)$$

for any sufficiently smooth function $v(x)$. Therefore [10]

$$\lim_{\varepsilon\to 0}\sup_{|z|<r} \mathrm{P}\left(\sup_{0\le t\le T} |x_\varepsilon(t) - \bar{x}(t)| > C\right) = 0$$

for any *C>0, r>0, T>0,* $x_\varepsilon(0) = \bar{x}(0) = x$ and $y(0) = y$. Besides, if is there exists such an isolated point $\hat{x}$ that $f_0(\hat{x}, y) \equiv 0$ and $f'(\hat{x}) < 0$ then this point is locally asymptotically stable equilibrium of the initial Markov dynamical system (2.3)-(2.4) for any realization of the given by equation (2.1) Markov process.

The second step of the stochastic approximation procedure is the analysis of the normal deviations

$$X_\varepsilon(t) = \frac{x_\varepsilon(t) - \bar{x}(t)}{\sqrt{\varepsilon}} \tag{2.10}$$

After substitution $x_\varepsilon(t) = \bar{x}(t) + \sqrt{\varepsilon} X_\varepsilon(t)$ in equation (2.3) - (2.4)

$$\tau_{k-1} < t < \tau_k, k \in N : \frac{d}{dt} X_\varepsilon(t) = \frac{F(\bar{x}(t) + \sqrt{\varepsilon} X_\varepsilon(t)) - f(\bar{x}(t))}{\sqrt{\varepsilon}} \tag{2.11}$$

$$t \in \mathrm{T} : X_\varepsilon(t) = X_\varepsilon(t-) + \frac{1}{\sqrt{\varepsilon}} G(\bar{x}(t) + \sqrt{\varepsilon} X_\varepsilon(t-), y_\varepsilon(t-)) \tag{2.12}$$

The equations (2.1) - (2.6) - (2.11) - (2.12) define three-dimensional Markov process on $\mathbf{R}^2 \times U$ with the weak infinitesimal operator

$$\begin{gathered} G(\varepsilon) v(\bar{x}, x, y) = f(\bar{x}) \frac{\partial}{\partial \bar{x}} v(\bar{x}, x, y) + \frac{F(\bar{x}(t) + \sqrt{\varepsilon} X_\varepsilon(t)) - f(\bar{x}(t))}{\sqrt{\varepsilon}} \frac{\partial}{\partial x} v(\bar{x}, x, y) + \\ + \varepsilon^{-1} Q v(\bar{x}, x, y) + \varepsilon^{-1} \lambda \left[v_\varepsilon(\bar{x}, x + \sqrt{\varepsilon} G(\bar{x}(t) + \sqrt{\varepsilon} X_\varepsilon(t-), y_\varepsilon(t-)), y) - v(\bar{x}, x, y) \right] \end{gathered} \tag{2.13}$$

To derive a formula for the weak infinitesimal operator of the approximative Markov process for normalized deviations we will apply the same as before. Let us consider $\{G(\varepsilon) v_\varepsilon, v_\varepsilon\}$, where

$$v_\varepsilon(\bar{x}, x, y) = v(\bar{x}, x) + \sqrt{\varepsilon} v_1(\bar{x}, x, y) + \varepsilon v_2(\bar{x}, x, y) \tag{2.14}$$

$v \in \mathbf{C}^2(\mathbf{R}^2)$ is an arbitrary positive function,

$$v_1(\bar{x}, x, y) = \Pi f_0(x, y) \frac{\partial}{\partial x} v(\bar{x}, x) \tag{2.15}$$

$$v_2(\bar{x}, x, y) = x \Pi f_0(x, y) \frac{\partial^2}{\partial x \partial \bar{x}} v(\bar{x}, x) + \lambda \frac{\Pi G^2(\bar{x}, y)}{2} \frac{\partial^2}{\partial x^2} v(\bar{x}, x) \tag{2.16}$$

After substitution (2.14) in (2.13) and decomposition accurate within $o\left(\sqrt{\varepsilon}\right)$ we can write the formula

$$\begin{aligned} (G(\varepsilon) v_\varepsilon(\bar{x}, x, y) = {} & f(\bar{x}) \frac{\partial}{\partial \bar{x}} v(\bar{x}, x) + x \frac{\partial}{\partial \bar{x}} \left[F(\bar{x}, y) + \lambda G(\bar{x}, y) \right] \frac{\partial}{\partial x} v(\bar{x}, x) + \\ & + \lambda \frac{G^2(\bar{x}, y)}{2} \frac{\partial^2}{\partial x^2} v(\bar{x}, x) + Q v_2(\bar{x}, x, y) + \\ & + \frac{1}{\sqrt{\varepsilon}} \left\{ \left[F(\bar{x}, y) + \lambda G(\bar{x}, y) - f(\bar{x}) \right] \frac{\partial}{\partial x} v(\bar{x}, x) + Q v_1(\bar{x}, x, y) \right\} + o\left(\sqrt{\varepsilon}\right) \end{aligned}$$

Now subject to the formulae (2.15) - (2.16) we can to ensure that

$$\lim_{\varepsilon \to 0} G(\varepsilon) v_\varepsilon(\bar{x}, x, y) = \hat{L} v(\bar{x}, x)$$

where

$$(\hat{L} v(\bar{x}, x) = f(\bar{x}) \frac{\partial}{\partial \bar{x}} v(\bar{x}, x) + x \frac{\partial}{\partial \bar{x}} f(\bar{x}) \frac{\partial}{\partial x} v(\bar{x}, x) + \lambda \frac{\sigma^2(\bar{x})}{2} \frac{\partial^2}{\partial x^2} v(\bar{x}, x) \tag{2.17}$$

and $\sigma^2(\bar{x}) = \int_U G^2(\bar{x}, y)\pi(dy)$. Operator $\hat{L}$ defines [5] a two-dimensional dynamical system composed of the ordinary differential equation (2.6) and the stochastic Ito differential equation

$$dX(t) = X(t)\, f'(\bar{x}(t))dt + \sigma(\bar{x}(t))\sqrt{\lambda}dw(t) \tag{2.18}$$

Thus for a sufficiently small $\varepsilon > 0$ for any initial conditions $x_\varepsilon(0) = x, y_\varepsilon(0) = y$ the finite dimensional distributions of the process $\{x_\varepsilon(t), t \ge 0\}$ given by equations (2.3)-(2.4) may be approximated [10,14] by the corresponding finite dimensional distributions of the process

$$\hat{x}_\varepsilon(\mathrm{t}) = \bar{x}(t) + \sqrt{\varepsilon}X(t) \tag{2.19}$$

where $\{\bar{x}(t), t \ge 0\}$ is the solution of equation (2.6) with the initial condition $\bar{x}(0) = x$ and where $\{X(t), t \ge 0\}$ is the solution of equation (2.18) with the initial condition $X(0) = 0$.

3 Stochastic approximation for logistic impulsive Markov model

Let us study a logistic type model with impulse type random perturbations given by the equations (1.5) - (1.6) and Poisson process with infinitesimal operator (2.1). This stochastic dynamical system may be defined as two-dimensional system of stochastic Poisson type differential equations

$$\begin{aligned} dx_\varepsilon(t) &= \lfloor r x_\varepsilon(t)(1 - K^{-1}x_\varepsilon(t)) - m x_\varepsilon(t) \rfloor dt + \\ &+ \varepsilon \int_U x_\varepsilon(t)\left[\alpha(y(t-)) + \beta(y(t-))x_\varepsilon(t-)\right]\mu(\varepsilon^{-1}\lambda, dt, du) \end{aligned} \tag{3.1}$$

$$dy = \int_U (u - y)\mu(\varepsilon^{-1}\lambda, dt, du) \tag{3.2}$$

where $\lambda \in \mathbf{R}_+, r > 0$, $m \ge 0, \varepsilon \in (0, \varepsilon_0)$ is a small positive parameter, $\mu(\varepsilon^{-1}\lambda, dt, du)$ is a Poisson measure on $U \subset \mathbf{R}_+ \times \mathbf{R}^m$ with parameter $\varepsilon^{-1}\lambda dt \pi(du) \pi(du)$ is a probabilistic measure on U, $\{\alpha(y), \beta(y)\}$ are bounded measurable functions on U,

$$\int_U \alpha(y)\pi(dy) = 0, \int_U \beta(y)\pi(dy) = 0, a = \int_U \alpha^2(y)\pi(dy), b = \int_U \beta^2(y)\pi(dy), \int_U \alpha(y)\beta(y)\pi(dy) = 0$$

Making use of stochastic averaging procedure we can derive an equation in the form of the classical logistic differential equation (1.2) for the mean of the population aggregate biomass:

$$\frac{d\bar{x}}{dt} = r\bar{x}(1 - K^{-1}\bar{x}) - m\bar{x} \tag{3.3}$$

For $\bar{r} = \bar{m}$ the solution of (3.3) is hyperbola

$$\bar{x}(t) = \frac{Kx(0)}{K + r\bar{x}(0)t} \tag{3.4}$$

and for $r \ne m$ it may be presented as follows:

$$\bar{x}(t) = \frac{x}{1 + xrK^{-1}(r - m)^{-1}\left(\exp\{(r - m)t\} - 1\right)}\exp\{(r - m)t\} \tag{3.5}$$

If $r < m$ then the function (3.5) monotonically decreases to zero. If $r \geq m$ then it increases to $\hat{x} = \frac{r-m}{rK^{-1}}$. Therefore [12] if the mean value of the birth rate is sufficiently lower than the mean value of the death rate, the population will almost surely go extinct, but for $r - m = O(\varepsilon)$ we may not apply an averaging procedure for asymptotic analysis of the equations (2.3) - (2.4).
Applying formula (2.18) we can approximate the finite dimensional distribution functions of the normalized deviations $X_\varepsilon(t) = \frac{x_\varepsilon(t) - \overline{x}(t)}{\sqrt{\varepsilon}}$ by the corresponding distribution functions of the solution of the stochastic differential equation

$$dX(t) = (r - m - 2rK^{-1}\overline{x}(t))X(t)dt + \sigma(\overline{x}(t))dw(t) \tag{3.6}$$

with the initial condition $X(0) = 0$. The above solution is the Gaussian process [5] with zero mean and variance that satisfies the equation

$$\frac{d}{dt}\mathrm{E}\{X^2(t)\} = 2(r - m - 2rK^{-1}\overline{x}(t))\mathrm{E}\{X^2(t)\} + \overline{x}^4(t)(a + b\overline{x}^2(t))^2\lambda \tag{3.7}$$

where $\sigma(x) = x^2(a + bx^2)\sqrt{\lambda}$. If $r > m$ then $\lim_{t\to\infty}\overline{x}(t) = \frac{r-m}{rK^{-1}} := \tilde{x}$ and for asymptotic analysis of the Markov process defined by the equation (3.6) we can simplify this equation as follows:

$$d\tilde{X}(t) = (m - r)\tilde{X}(t)dt + \sigma(\tilde{x})dw(t) \tag{3.8}$$

The stationary diffusion process $\hat{X}(t) = \sigma(\tilde{x})\int_{-\infty}^{t}\exp\{(m-r)(t-s)\}dw(s)$ satisfies the equation (3.8), has zero mean, the ergodic distribution $N\left(0, \frac{\sigma^2(\tilde{x})}{2(r-m)}\right)$, and has the covariance function $K(t,\tau) = \mathrm{E}\{\hat{X}(t)\hat{X}(\tau)\} = \frac{\sigma^2(\tilde{x})}{2(r-m)}\exp\{2(m-r)(t-\tau)\}$. The difference of any other solution $\tilde{X}(t)$ of (3.8) on $\hat{X}(t)$ exponentially tends to zero.

Conclusion

Stochastic approximation procedure provides a useful and informative way to explore a logistic type population mathematical model given in a form of Markov impulsive differential equation with discontinuities in a form of a fast oscillating ergodic Poisson process. This method allows to analyse the ordinary differential equation describing the mean value of population size and stochastic Ito differential equation for random deviations. Probabilistic characteristics of the resulting processes are found in explicit form.

References

[1] BAZYKIN, A.D., *Nonlinear dynamics of the interacting populations*. Moscow, IKI, 2003 (rus.).

[2] SVIREZHEV, Yu.M., LOGOPHED, D.J., *Stability of biological community*. Moscow, Nauka, 1978, (rus.)

[3] KHODABIN, M., KIAEE, N., Stochastic dynamical logistic population growth model. *Journal of Mathematical Sciences: Advances and Applications*, vol. 11, Nr. 1, 2011, pp. 11-29.

[4] TUCKWELL, H., KKOZIOL, J., Logistic population growth under random dispersal, *Bulletin of Mathematical Biology* Vol. 49, No. 4, pp. 495-506, 1987.

[5] GIKHMAN, I.I. , SKOROKHOD, A.V., *Stochastic Differential Equations,* Springer-Verlag, 1972.

[6] MAY, R., *Stability and Complexity in Model Ecosystems.* Princeton, NJ: Princeton University Press, 1973.

[7] LUDWIG, D. 1975. Persistence of Dynamical Systems under Random Perturbations. *SIAM Rev.* 17, 605-640.

[8] MURRAY, J., *Mathematical Biology I: An Introduction*, volume I. Springer-Verlag, 3rd edition, 2003.

[9] DYNKIN, Ye.B. *Theory of Markov Processes.* Prentice-Hall. 1961.

[10] SKOROKHOD, A.V. *Asymptotic methods in the theory of stochastic differential equations* (Transl. Math. Monographs 78), American Mathematical Society, Providence, RI, 1989.

[11] CARKOVS, J., ŠADURSKIS, K. Dynamics of a population subject to impulse type random loss. In *Proceedings of the Latvian Academy of sciences*. Section B, Vol. 71 (2017), No. 4 (709), pp. 298–302.

[12] TSARKOV, Ye.F., (CARKOVS, J.), Asymptotic methods for stability analysis Markov impulse dynamical systems, *Nonlinear dynamics and system theory*, 9 (2002), No. 2(1), 103–115.

Current address

Carkovs Jevgeņijs, prof, Dr.hab.math.
Department of Probability Theory and Mathematical Statistics
Riga Technical University
Kaļķu iela 1, LV-1658, Riga, Latvia
E-mail: carkovs@livas.lv

Šadurskis Kārlis, prof., Dr.math.
Department of Probability Theory and Mathematical Statistics
Riga Technical University
Kaļķu iela 1, LV-1658, Riga, Latvia
E-mail: karlis.sadurskis@gmail.com

Goldšteine Jolanta, doc, Dr.math.
Department of Probability Theory and Mathematical Statistics
Riga Technical University
Kaļķu iela 1, LV-1658, Riga, Latvia
E-mail: Jolanta.goldsteine@lezards.lv